Tutorial Week 2

Topics: metrics, topologies, continuous functions.

2.1. Let X be a set and d the discrete metric on X, that is $d(x_1, x_2) = 1$ for all $x_1 \neq x_2$; see also Exercise 1.5. Prove that the topology defined by d is the discrete topology.

2.2. Is the word "finite" necessary in the statement of Proposition 2.10? If no, give a proof of the statement without "finite". If yes, give an example of a metric space (X, d) and an infinite collection of open subsets of X whose intersection is not an open set.

2.3. Find all topologies on the set $\{0, 1\}$ and determine which of them are metrisable.

2.4. Let X be a set and S a subset of $\mathcal{P}(X)$. Prove that the topology generated by S is the intersection of all topologies $\mathcal{T}$ on X containing S, and is thus the coarsest among such topologies.

2.5. Let X and Y be two topological spaces, where the topology on X is the discrete topology. Prove that every function from X to Y is continuous.

2.6. Let A be a subset of a topological space X. Prove that

(a) $\partial A \cap A^\circ = \varnothing$;

(b) $\overline{A} = A^\circ \cup \partial A$;

(c) $A^\circ = A \smallsetminus \partial A$.

2.7. Let $f\colon X \longrightarrow Y$ be a function and $\mathcal{T}_X$ a topology on X. Define

$$\mathcal{T}_Y = \left\{U \in \mathcal{P}(Y) : f^{-1}(U) \in \mathcal{T}_X\right\}.$$

(a) Prove that $\mathcal{T}_Y$ is the finest topology on Y such that f is continuous. (This topology is called the *final topology* induced by f.)

(b) Let $\mathcal{T}$ be another topology on Y. Prove that $f\colon (X, \mathcal{T}_X) \longrightarrow (Y, \mathcal{T})$ is continuous if and only if $\mathcal{T}$ is coarser than $\mathcal{T}_Y$.

Note: There is a "dual" setting where you start with a topology on Y and look for the coarsest topology on Y such that f is continuous, see Exercise 1.23.

2.8. Prove that a function $f\colon X \longrightarrow Y$ between metric spaces is continuous if and only if it satisfies the usual ε–δ definition: for every point x of X and every positive real number ε, there exists a positive real number δ such that $d_X(x, y) < \delta$ implies $d_Y\big(f(x), f(x)\big) < \varepsilon$.

2.9.

(a) Let $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$ be functions, where X, Y, Z are sets, and let $S \subseteq Z$. Then

$$f^{-1}\big(g^{-1}(S)\big) = (g \circ f)^{-1}(S).$$

(b) Let $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$ be continuous functions, where X, Y, Z are topological spaces. Prove that $g \circ f\colon X \longrightarrow Z$ is continuous.