Tutorial Week 3

Topics: Closure, interior, denseness, product, Hausdorff, equivalent metrics, disconnectedness

3.1. Let A and B be subsets of a topological space X.

(a) Suppose $A \subseteq B$. Prove that $\overline{A} \subseteq \overline{B}$ and $A^\circ \subseteq B^\circ$.

(b) Prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$ and $(A \cap B)^\circ = A^\circ \cap B^\circ$.

(c) Prove that $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$. Find an example in which $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$.

(d) Prove that $(A \cup B)^\circ \supseteq A^\circ \cup B^\circ$. Find an example in which $(A \cup B)^\circ \neq A^\circ \cup B^\circ$.

[**Hint**: For (c) and (d), think of some subsets of $\mathbf{R}$.]

3.2. A subset $D \subseteq X$ of a topological space X is dense in X if and only if $D \cap U \neq \varnothing$ for all nonempty open sets U in X.

3.3. Let X be a topological space. The intersection of two dense open sets U_1 and U_2 is dense and open.

3.4. Let (X, d) be a metric space and let $A \subseteq X$.

(a) Prove that the set A is open if and only if it is the union of a collection of open balls.

(b) Conclude that the set of all open balls in X generates the metric topology of X.

3.5. Let (X, d) be a metric space.

(a) Prove that the metric topology on (X, d) is generated by open balls of radii smaller than 1.

(b) Define $d' \colon X \times X \longrightarrow \mathbf{R}_{\geqslant 0}$ by

$$d'(x, y) = \min\left\{d(x, y), 1\right\}.$$

Prove that d' is a metric.

(c) Prove that d and d' are equivalent (that is, they give rise to the same topology on X).

3.6. Let C_1 and C_2 be two connected subsets of a topological space X such that $C_1 \cap C_2 \neq \varnothing$. Prove that $C_1 \cup C_2$ is connected.

3.7. Let X be a topological space and define $x \sim x'$ if there exists a connected subset $C \subseteq X$ such that $x, x' \in C$. Prove that this is an equivalence relation on the set X.
(The equivalence classes are called the *connected components* of X).

3.8. Let X, Y_1, and Y_2 be topological spaces, and $\pi_1 \colon Y_1 \times Y_2 \longrightarrow Y_1$ and $\pi_2 \colon Y_1 \times Y_2 \longrightarrow Y_2$ be the projections. Prove that a function $f \colon X \longrightarrow Y_1 \times Y_2$ is continuous if and only if both $\pi_1 \circ f$ and $\pi_2 \circ f$ are continuous.

3.9. Given a set X, define the *diagonal function*

$$\Delta \colon X \longrightarrow X \times X, \qquad x \longmapsto (x, x).$$

(a) Prove that two subsets A and B of X are disjoint if and only if $\Delta(X)$ and $A \times B$ are disjoint.

(b) If X is a topological space, prove that Δ is continuous.

(c) Prove that a topological space X is Hausdorff if and only if $\Delta(X)$ is closed in $X \times X$.