Tutorial Week 4

Topics: Closed functions, compactness, sequences

4.1. Let X, Y be topological spaces. Recall that a function $f\colon X \longrightarrow Y$ is said to be *closed* if for every closed subset $C \subseteq X$, the image $f(C)$ is closed in Y.

(a) Prove that the composition of two closed maps is a closed map.

(b) Prove that a continuous bijection between topological spaces is a homeomorphism if and only if it is closed.

(c) Give an example of a bijection $f\colon X \longrightarrow Y$ between topological spaces such that f is continuous but not closed (and therefore f^{-1} is closed but not continuous).

4.2. Let X, Y be topological spaces. Recall that a function $f\colon X \longrightarrow Y$ is said to be *open* if for every open set $U \subseteq X$, the image $f(U)$ is open in Y.

(a) Give an example of a function that is open but not closed.

(b) Give an example of a function that is closed but not open.

4.3. Let K and L be compact subsets of a topological space X. Prove that $K \cup L$ is compact.

4.4. Prove that every finite topological space is compact.

4.5. Prove that a discrete topological space X is compact if and only if X is finite.

4.6. Let X be a compact topological space and let Y be a Hausdorff topological space. Prove that every continuous bijection from X to Y is a homeomorphism.

4.7. Let (x_n) be a sequence in a metric space X, let $\varphi\colon \mathbf{N} \longrightarrow \mathbf{N}$ be an injective function, and consider the sequence $(y_n) = \big(x_{\varphi(n)}\big)$ in X. Prove that if (x_n) converges to x, then so does (y_n).

Does the converse hold?

4.8. Give $\mathbf{N} \subseteq \mathbf{R}$ the subspace topology. Let X be a topological space and (x_n) a sequence in X. Prove that (x_n) is a continuous function $\mathbf{N} \longrightarrow X$.

4.9. Let (X, d_X) and (Y, d_Y) be metric spaces and let d be the sup norm metric on $X \times Y$:

$$d\big((x_1, y_1), (x_2, y_2)\big) = \max\big(d_X(x_1, x_2), d_Y(y_1, y_2)\big).$$

Prove that $\big((x_n, y_n)\big) \longrightarrow (x, y) \in X \times Y$ if and only if $(x_n) \longrightarrow x \in X$ and $(y_n) \longrightarrow y \in Y$.