Tutorial Week 5

Topics: Sequences, completeness, uniform continuity

5.1. Let d_1 and d_2 be equivalent metrics (they define the same topology) on a set X. Prove that a sequence converges to a point x in (X, d_1) if and only if it converges to x in (X, d_2).

5.2. Let $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$ be uniformly continuous functions between metric spaces. Prove that $g \circ f\colon X \longrightarrow Z$ is uniformly continuous.

5.3. Let $f\colon X \longrightarrow Y$ be a uniformly continuous function between two metric spaces and suppose $(x_n) \sim (x'_n)$ are equivalent sequences in X. Prove that $\big(f(x_n)\big) \sim \big(f(x'_n)\big)$ as sequences in Y.

Does the conclusion hold if f is only assumed to be continuous?

5.4. Let X be a complete metric space and let $S \subseteq X$. Prove that the closure $\overline{S}$ (with the metric induced from $\overline{S} \subseteq X$) is a completion of S (with the metric induced from $S \subseteq X$).

5.5. Let (X, d_X) and (Y, d_Y) be metric spaces and $f\colon X \longrightarrow Y$ a surjective continuous function. Suppose that X is complete and for all $x_1, x_2 \in X$ we have

$$d_X\big(x_1, x_2\big) \leqslant d_Y\big(f(x_1), f(x_2)\big).$$

Prove that Y is complete.

In particular, isometries preserve completeness.

5.6. Let (X, d) be a metric space and let $S \subseteq X$ be a nonempty subset. Define $d_S\colon X \longrightarrow \mathbf{R}_{\geqslant 0}$ by

$$d_S(x) = \inf_{s \in S} d(x, s).$$

(a) Prove that d_S is uniformly continuous.

[**Hint**: Show that $|d_S(x) - d_S(y)| \leqslant d(x, y)$ for all $x, y \in X$.]

(b) Prove that $d_S(x) = 0$ if and only if $x \in \overline{S}$.

(c) Prove that if $U \subseteq X$ is an open neighbourhood of x, then $d_{X \setminus U}(x) > 0$.