Tutorial Week 6

Topics: Contractions, Banach fixed point theorem

6.1. Prove that any contraction is uniformly continuous.

6.2. Consider the equation

$$x^3 - x - 1 = 0. \tag{1}$$

(a) Show that the equation must have **at least one solution** in the interval $[1, 2]$.

(b) Show that the function $f\colon [1,2] \longrightarrow \mathbf{R}$ given by

$$f(x) = (1+x)^{1/3}$$

has image contained in $[1, 2]$ and is a contraction.

(c) Show that Equation (1) has **a unique solution** ξ in the interval $[1, 2]$ and describe a sequence of real numbers that converges to ξ.

6.3. Find a non-empty metric space X and a contraction $f\colon X \longrightarrow X$ such that f has no fixed points.

6.4. Recall Newton's method for solving equations: given a differentiable function g and an initial guess x_0, iterate

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}, \qquad n \geqslant 0.$$

The aim is to get a sequence (x_n) that converges to a root of g.

Apply this to the function $g(x) = x^2 - 3$:

(a) Prove that $f(x) := x - g(x)/g'(x)$ defines a contraction from $X = [\sqrt{3}, \infty)$ to itself.

(b) Use the Banach Fixed Point Theorem to conclude that the Newton iteration converges to $\sqrt{3}$ for any starting point $x_0 \in X = [\sqrt{3}, \infty)$.

(c) What happens if we pick a starting point $x_0 \in (0, \sqrt{3})$?

6.5. Let $A = (a_{ij})$ be an $n \times n$ real matrix with all $|a_{ij}| < 1$.

Given a nonzero real eigenvalue λ of A, consider the function $f_\lambda\colon \mathbf{R}^n \longrightarrow \mathbf{R}^n$ given by

$$f_\lambda(v) = \frac{1}{\lambda} Av.$$

(a) Prove that if $|\lambda| \geqslant n$ then f_λ is a contraction for the sup metric d on $\mathbf{R}^n$:

$$d(x,y) = \max_{i \in \{1,\dots,n\}} |x_i - y_i|, \qquad x = \begin{pmatrix} x_1 & \dots & x_n \end{pmatrix}^\mathsf{T}, y = \begin{pmatrix} y_1 & \dots & y_n \end{pmatrix}^\mathsf{T} \in \mathbf{R}^n.$$

(b) Use the Banach Fixed Point Theorem to derive a contradiction, and thus conclude that every real eigenvalue λ of A satisfies $|\lambda| < n$.