Tutorial Week 7

Topics: (Total) boundedness, uniform convergence

7.1. Prove that in any metric space (X, d), any totally bounded set S is bounded.

7.2. Find a bounded subset of a metric space that is not totally bounded.

7.3. Let (X, d) be a metric space.
Prove that if A and B are bounded sets with $A \cap B \neq \varnothing$, then

$$\operatorname{diam}(A \cup B) \leqslant \operatorname{diam}(A) + \operatorname{diam}(B).$$

What happens if $A \cap B = \varnothing$?

7.4.

(a) Prove that every subspace of a totally bounded space is totally bounded.

(b) Suppose a metric space X has a totally bounded dense subset D. Prove that X is totally bounded.

(c) Prove that a metric space X is totally bounded if and only if it is isometric to a subspace of a compact metric space. [**Hint**: Completion.]

7.5. We say that a topological space is *separable* if it contains a countable dense subset. (Easy examples are $\mathbf{R}$ with countable dense subset $\mathbf{Q}$, or more generally $\mathbf{R}^n$ with countable dense subset $\mathbf{Q}^n$.)
Prove that any totally bounded metric space X is separable.

7.6. Given metric spaces X, Y, prove that a sequence (f_n) in $B(X, Y)$ converges uniformly to $f \in B(X, Y)$ if and only if $(f_n) \longrightarrow f$ with respect to the uniform metric d_∞ on $B(X, Y)$.

7.7. Give an example of a sequence of bounded continuous functions that converges pointwise to a discontinuous function.
[**Hint**: Consider the behaviour of x^n as $n \longrightarrow \infty$.]