# Tutorial Week 8

**Topics:** Pointwise and uniform convergence, approximation, Baire.

**8.1.** For each $n \in \mathbf{N}$, consider the function $f_n \colon [0,1] \longrightarrow \mathbf{R}$ given by

$$f_n(x) = \frac{x^2}{1+nx}.$$

(a) Prove that $f_n$ is bounded, for all $n \in \mathbf{N}$.

(b) Find the pointwise limit $f$ of the sequence $(f_n)$.

(c) For any $n \in \mathbf{N}$, compute the uniform distance $d_\infty(f_n, f)$.

(d) Does the sequence $(f_n)$ converge uniformly to $f$?

**8.2.** Let $f_0 \colon \mathbf{R} \longrightarrow \mathbf{R}$ be the function defined by

$$f_0(x) = \begin{cases} 1+x & \text{if } -1 \leqslant x \leqslant 0, \\ 1-x & \text{if } 0 < x \leqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

For each positive integer $n$, define $f_n \colon \mathbf{R} \longrightarrow \mathbf{R}$ by

$$f_n(x) = f_0(x-n).$$

(a) Prove that $f_n$ is bounded, for all $n \in \mathbf{N}$.

(b) Find the pointwise limit $f$ of the sequence $(f_n)$.

(c) For any $n \in \mathbf{N}$, compute the uniform distance $d_\infty(f_n, f)$.

(d) Does the sequence $(f_n)$ converge uniformly to $f$?

**8.3.** Let $X = [0,1] \times [0,1]$ be the unit square with the induced topology from $\mathbf{R}^2$.
Find a subalgebra $\mathcal{A}$ of $C_0(X, \mathbf{R})$ that is dense. (Obviously, try to make $\mathcal{A}$ as small as you can.)

**8.4.**

(a) Suppose $f \in C_0([0,1], \mathbf{R})$ has the property that

$$\int_0^1 f(x)\, x^n \, dx = 0 \qquad \text{for all } n = 0, 1, 2, \dots.$$

Prove that $f$ is the constant function $0$ on $[0,1]$.

(b) Give an explicit **discontinuous** function $f \colon [0,1] \longrightarrow \mathbf{R}$ that satisfies the equation in part (a) but is (obviously) not the constant function $0$ on $[0,1]$.

**8.5.** Let $(X, d)$ be a nonempty complete metric space. If

$$X = \bigcup_{n \in \mathbf{N}} C_n \quad \text{with each } C_n \text{ a closed subset of X,}$$

then there exists $n \in \mathbf{N}$ such that $C_n^\circ \neq \varnothing$.
[**Hint**: Use the Baire Category Theorem, Theorem 2.79.]