Tutorial Week 9

Topics: Norms, inner products

9.1. Let V be an inner product space. Prove that for any $v \in V$ we have

$$\|v\| = \sup_{\|w\|=1} \left|\langle v, w\rangle\right|.$$

Show that the supremum is in fact achieved by a well-chosen w.

9.2 (Pythagorean theorem)**.** Let $v_1, \dots, v_n$ be pairwise orthogonal vectors in an inner product space V. Prove that

$$\|v_1 + \cdots + v_n\|^2 = \|v_1\|^2 + \cdots + \|v_n\|^2.$$

9.3. Let $(V, \|\cdot\|)$ be a normed space.

(a) Fix $u \in V$ and define $T_u \colon V \longrightarrow V$ by $T_u(v) = u + v$ for all $v \in V$. Prove that T_u is a bijective isometry.

(b) Fix $\alpha \in \mathbf{F}^\times$ and define $S_\alpha \colon V \longrightarrow V$ by $S_\alpha(v) = \alpha v$ for all $v \in V$. Prove that S_α is a homeomorphism.

(c) Let $U_1, \dots, U_n$ be nonempty open subsets of V. Let $\alpha_1, \dots, \alpha_n \in \mathbf{F}$ such that at least one $\alpha_i \neq 0$. Let

$$U = \alpha_1 U_1 + \cdots + \alpha_n U_n.$$

Prove that U is a nonempty open subset of V.

9.4. Prove that the following norms on $\mathbf{R}^n$ are not defined by inner products:

(a) the ℓ^1-norm defined by

$$\left\|(x_1, \dots, x_n)\right\|_1 = \sum_{i=1}^{n} |x_i|,$$

(b) the ℓ^∞-norm defined by

$$\left\|(x_1, \dots, x_n)\right\|_\infty = \max\left\{|x_1|, \dots, |x_n|\right\}.$$

9.5. Let $\left(V, \|\cdot\|\right)$ be a Banach space. Prove that if a series $\sum_{n=1}^{\infty} a_n$ in V converges absolutely, then

$$\left\|\sum_{n=1}^{\infty} a_n\right\| \leqslant \sum_{n=1}^{\infty} \|a_n\|.$$

(If V is not complete, the result still holds under the extra assumption that the series **converges** in V.)

9.6. Prove that every finite-dimensional normed vector space V is separable.
More precisely, fix a basis $\{v_1, \dots, v_n\}$ of V and let

$$\mathbf{K} = \begin{cases} \mathbf{Q} & \text{if } \mathbf{F} = \mathbf{R} \\ \mathbf{Q}[i] & \text{if } \mathbf{F} = \mathbf{C} \end{cases} \qquad \text{and } D = \operatorname{Span}_{\mathbf{K}}\{v_1, \dots, v_n\}.$$

Prove that D is a dense countable subset of V.