Tutorial Week 10

Topics: bounded linear transformations, convexity, sequence spaces.

10.1. Let $X = C_0\big([0,1], \mathbf{R}\big)$ be the Banach space of continuous functions

$$f\colon [0,1] \longrightarrow \mathbf{R}$$

with the supremum norm, as described in Proposition 3.13.

Define $\phi\colon X \longrightarrow \mathbf{R}$ by $\phi(f) = f(0)$ for all $f \in X$.

Prove that ϕ is a bounded linear map.

10.2. Let V, W be normed spaces over $\mathbf{F}$. Let C be a convex subset of V and D a convex subset of W. Prove that $C \times D$ is a convex subset of $V \times W$.

10.3 (Minkowski's Inequality for ℓ^∞)**.** Prove that if $u, v \in \ell^\infty$, then

$$\|u + v\|_{\ell^\infty} \leqslant \|u\|_{\ell^\infty} + \|v\|_{\ell^\infty}.$$

In particular, $u + v \in \ell^\infty$.

10.4 (Hölder's Inequality for ℓ^1 and ℓ^∞)**.** Prove that if $u = \big(u_n\big) \in \ell^\infty$ and $v = \big(v_n\big) \in \ell^1$, then

$$\sum_{n=1}^{\infty} |u_n v_n| \leqslant \|u\|_{\ell^\infty} \|v\|_{\ell^1}.$$

10.5. Suppose $1 \leqslant p \leqslant q$. Prove that

$$\ell^p \subseteq \ell^q.$$

Show that if $p < q$ then the inclusion is strict: $\ell^p \subsetneq \ell^q$.

10.6.

(a) Prove that the function $f\colon \ell^1 \longrightarrow \mathbf{F}$ defined by

$$f\big((a_n)\big) = \sum_{n=1}^{\infty} a_n.$$

is continuous.

(b) Prove that the following subset is a closed subspace of ℓ^1:

$$S = \left\{ (a_n) \in \ell^1 : \sum_{n=1}^{\infty} a_n = 0 \right\}.$$

10.7. Consider the left shift map $L\colon \mathbf{F}^{\mathbf{N}} \longrightarrow \mathbf{F}^{\mathbf{N}}$ given by $L\big((a_n)\big) = (a_{n+1})$, that is

$$L(a_1, a_2, a_3, \dots) = (a_2, a_3, \dots).$$

(a) Prove that L is a surjective linear map. What is the kernel of L?

(b) Prove that for all $1 \leqslant p \leqslant \infty$, the restriction of L to ℓ^p is a surjective continuous map onto ℓ^p.

(c) Define the right shift map $R\colon \mathbf{F}^{\mathbf{N}} \longrightarrow \mathbf{F}^{\mathbf{N}}$ and prove that it is an injective linear map, the restriction of which is an isometry for any ℓ^p with $1 \leqslant p \leqslant \infty$.

(d) Check that $L \circ R = \mathrm{id}_{\mathbf{F}^{\mathbf{N}}} \neq R \circ L$.