Tutorial Week 11

Topics: Orthogonality and projections in Hilbert spaces.

11.1. Let H be a Hilbert space. Complete the proof of Corollary 3.37 by showing that:

(a) If V is a linear subspace of H, then $\left(V^{\perp}\right)^{\perp} = \overline{V}$.

(b) If S is a subset of H, then $\left(S^{\perp}\right)^{\perp} = \overline{\operatorname{Span}(S)}$.

11.2. Let S be a subset of a Hilbert space H. Prove that $\operatorname{Span}(S)$ is dense in H if and only if $S^{\perp} = 0$.
[**Hint**: Use Corollary 3.37.]

11.3. Let H be a Hilbert space and let $\pi\colon H \longrightarrow H$ be a projection. Prove that π is an orthogonal projection if and only if $\mathrm{id}_H - \pi$ is an orthogonal projection.
[**Hint**: Use Corollary 3.37.]

11.4. Fix $j \in \mathbf{N}$ and consider the map $\pi_j\colon \mathbf{F}^{\mathbf{N}} \longrightarrow \mathbf{F}$ given by

$$\pi_j\big((a_n)\big) = a_j.$$

(a) Show that π_j is linear.

(b) Prove that the restriction of π_j to ℓ^p for $1 \leqslant p \leqslant \infty$ is bounded (hence continuous) and surjective.

11.5. Let $H = \ell^2$ over $\mathbf{R}$ and consider the subset

$$W = \left\{y = (y_n) \in \ell^2 \colon y_n \geqslant 0 \text{ for all } n \in \mathbf{N}\right\}.$$

(a) Prove that W is a closed, convex subset of H. Is it a vector subspace?

[**Hint**: Use Tutorial Question 11.4 to prove that W is closed.]

(b) Find the closest point $y_{\min} \in W$ to

$$x = (x_n) = \left(\frac{(-1)^n}{n}\right) = \left(-1, \frac{1}{2}, -\frac{1}{3}, \dots\right)$$

and compute $d_W(x)$.

[**Hint**: You may use without proof the identity $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.]

11.6. Let φ be a nonzero orthogonal projection (that is, φ is not the constant function 0) on an inner product space V. Prove that $\|\varphi\| = 1$.
[**Hint**: Fix $x \in V$ and show that $x - \varphi(x) \in \ker\varphi$, then that $\|\varphi(x)\|^2 = \langle x, \varphi(x)\rangle$.]