Tutorial Week 12

Topics: adjoints.

12.1. Let $R\colon \ell^2 \longrightarrow \ell^2$ and $L\colon \ell^2 \longrightarrow \ell^2$ be the operators defined by

$$R(x_1, x_2, x_3, \dots) = (0, x_1, x_2, \dots) \quad \text{and} \quad L(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots)$$

Find the adjoints of R and L and prove that neither R nor L is normal.

12.2. Equip $\mathbf{R}^n$ with the standard Euclidean inner product and let $f\colon \mathbf{R}^n \longrightarrow \mathbf{R}^n$ be a linear transformation with standard matrix representation A.

(a) Prove that the adjoint $f^*\colon \mathbf{R}^n \longrightarrow \mathbf{R}^n$ has standard matrix representation A^T, the transpose of A.

(b) Prove that f is self-adjoint if and only if A is symmetric.

12.3. Let $f \in L(H)$ with H a Hilbert space. Then the maps

$$p = f^* \circ f \qquad \text{and} \qquad s = f + f^*$$

are self-adjoint.

12.4. Let H be a Hilbert space and let $f \in L(H)$ be a self-adjoint operator.

(a) Prove that $\langle f(x), x \rangle \in \mathbf{R}$ for all $x \in H$.

(b) Prove that all eigenvalues of f are real.

12.5. The composition of two self-adjoint maps on a Hilbert space is self-adjoint if and only if the maps commute.

12.6. Let $(u_i)_{i \in I}$ be an orthonormal basis of an inner product space V and let $v \in V$. Prove that $v = 0$ if and only if $\langle v, u_i \rangle = 0$ for every index $i \in I$.

12.7. Consider the Hilbert space ℓ^2 of square-summable complex sequences $(a_1, a_2, \dots)$.
Let (λ_n) be a bounded sequence of complex numbers and define $T\colon \ell^2 \longrightarrow \ell^2$ by

$$T(a_1, a_2, \dots) = (\lambda_1 a_2, \lambda_2 a_4, \dots, \lambda_n a_{2n}, \dots).$$

(a) Show that T is a continuous linear operator.

(b) Compute the norm $\|T\|$.

(c) Find the adjoint operator T^*.