Tutorial Week 2

Topics: metrics, topologies, continuous functions.

2.1. Let X be a set and d the discrete metric on X, that is $d(x_1, x_2) = 1$ for all $x_1 \neq x_2$; see also Exercise 1.5. Prove that the topology defined by d is the discrete topology.

Solution. By the definition of the discrete metric, we have $\mathbf{B}_1(x) = \{x\}$ for every element x of X. If S is a subset of X, then

$$S = \bigcup_{x \in S} \{x\} = \bigcup_{x \in S} \mathbf{B}_1(x),$$

and thus S is open. Therefore, every subset of X is open in (X, d); in other words, the topology defined by d is the discrete topology. □

2.2. Is the word "finite" necessary in the statement of Proposition 2.10? If no, give a proof of the statement without "finite". If yes, give an example of a metric space (X, d) and an infinite collection of open subsets of X whose intersection is not an open set.

Solution. The word "finite" is necessary. For a counterexample to the more general statement, for each $n \in \mathbf{Z}_{\geqslant 1}$ take $U_n = (-1/n, 1/n)$ as an open set in $\mathbf{R}$ with the Euclidean metric. I claim that

$$U := \bigcap_{n \in \mathbf{Z}_{\geqslant 1}} U_n = \{0\}.$$

This can be proved by contradiction: suppose $u \in U$, $u \neq 0$. Let $n \in \mathbf{Z}_{\geqslant 1}$ be such that $n \geqslant \frac{1}{|u|}$. Then $|u| \geqslant \frac{1}{n}$, therefore $u \notin (-1/n, 1/n) = U_n$, contradiction.

Finally, U is not open: for any $r \in \mathbf{R}_{>0}$, $\frac{r}{2} \in \mathbf{B}_r(0)$ but $\frac{r}{2} \notin \{0\} = U$, so $\mathbf{B}_r(0)$ is not a subset of U. □

2.3. Find all topologies on the set $\{0, 1\}$ and determine which of them are metrisable.

Solution. Let $\mathcal{T}$ be a topology on $\{0, 1\}$. Since $\varnothing$ and $\{0, 1\}$ must belong to $\mathcal{T}$, there are four possibilities:

- $\mathcal{T} = \mathcal{P}(\{0, 1\})$. This is the discrete topology on $\{0, 1\}$.
- $\mathcal{T} = \{\varnothing, \{1\}, \{0, 1\}\}$. Since $\mathcal{T}$ is finite, it suffices to verify it is closed under binary intersection and binary union. We can prove this by enumeration:

  $$\varnothing \cap \{1\} = \varnothing, \qquad \varnothing \cap \{0,1\} = \varnothing, \qquad \{1\} \cap \{0,1\} = \{1\},$$
  $$\varnothing \cup \{1\} = \{1\}, \qquad \varnothing \cup \{0,1\} = \{0,1\}, \qquad \{1\} \cup \{0,1\} = \{0,1\}.$$

  Therefore, $\mathcal{T}$ is a topology.
- $\mathcal{T} = \{\varnothing, \{0\}, \{0, 1\}\}$. This can be proved to be a topology similarly.
- $\mathcal{T} = \{\varnothing, \{0, 1\}\}$. This is the trivial topology on $\{0, 1\}$.

Now suppose d is a metric on $\{0, 1\}$. Since $d(0, 1) > 0$, we can pick a positive real number r smaller than $d(0, 1)$. It follows that

$$\mathbf{B}_r(0) = \{0\} \quad \text{and} \quad \mathbf{B}_r(1) = \{1\},$$

and the metric topology defined by d is thus the discrete topology. Therefore, the only metrisable topology on $\{0, 1\}$ is the discrete topology. □

2.4. Let X be a set and S a subset of $\mathcal{P}(X)$. Prove that the topology generated by S is the intersection of all topologies $\mathcal{T}$ on X containing S, and is thus the coarsest among such topologies.

Solution. Let $\mathcal{T}_S$ be the topology generated by S and $\mathcal{T}_S'$ the intersection of all topologies $\mathcal{T}$ on X containing S.

We start with proving $\mathcal{T}_S'$ is a topology:

- Both $\varnothing$ and X belong to all topologies containing S, and thus belong to the intersection $\mathcal{T}_S'$.
- If $\{U_i \in \mathcal{T}_S' : i \in I\}$ is a collection of members of $\mathcal{T}_S'$, then $U_i \in \mathcal{T}$ for every $i \in I$ and every topology $\mathcal{T}$ containing S. It follows that $\bigcup_{i\in I} U_i \in \mathcal{T}$ for every topology $\mathcal{T}$ containing S, and thus $\bigcup_{i\in I} U_i \in \mathcal{T}_S'$.
- If $U_1, \dots, U_n$ are members of $\mathcal{T}_S'$, then they belong to every topology $\mathcal{T}$ containing S. It follows that $\bigcap_{i=1}^n U_i \in \mathcal{T}$ for every topology $\mathcal{T}$ containing S, and thus $\bigcap_{i=1}^n U_i \in \mathcal{T}_S'$.

It follows from the definition of $\mathcal{T}_S$ that $S \subseteq \mathcal{T}_S$, so $\mathcal{T}_S$ is finer than $\mathcal{T}_S'$. However, for $\mathcal{T}_S'$ to be a topology, it has to be closed under arbitrary union and finite intersection, and thus contains all members of $\mathcal{T}_S$; in other words, $\mathcal{T}_S'$ has to be finer than $\mathcal{T}_S$. Hence $\mathcal{T}_S' = \mathcal{T}_S$. □

2.5. Let X and Y be two topological spaces, where the topology on X is the discrete topology. Prove that every function from X to Y is continuous.

Solution. Consider a function $f : X \longrightarrow Y$. Since the topology on X is discrete, it follows that $f^{-1}(U)$ is open for every open subset U of Y, and thus f is continuous. □

2.6. Let A be a subset of a topological space X. Prove that

(a) $\partial A \cap A^\circ = \varnothing$;

(b) $\overline{A} = A^\circ \cup \partial A$;

(c) $A^\circ = A \smallsetminus \partial A$.

Solution. (a) $\partial A \cap A^\circ = \overline{A} \cap \overline{(X \smallsetminus A)} \cap A^\circ = \overline{(X \smallsetminus A)} \cap A^\circ$ since $A^\circ \subseteq A \subseteq \overline{A}$. Suppose $x \in \overline{(X \smallsetminus A)} \cap A^\circ$. By Proposition 2.19 every open neighbourhood of x intersects $X \smallsetminus A$ nontrivially; in particular A° intersects $X \smallsetminus A$ nontrivially, contradiction.

(b) Since $A^\circ \subseteq A \subseteq \overline{A}$ and $\partial A = \overline{A} \cap \overline{(X \smallsetminus A)} \subseteq \overline{A}$, the inclusion $A^\circ \cup \partial A \subseteq \overline{A}$ is clear.

In the other direction, let $x \in \overline{A}$ and suppose $x \notin \partial A$, which forces $x \notin \overline{(X \smallsetminus A)}$. By Proposition 2.19 there exists an open neighbourhood U_x of x such that $U_x \cap (X \smallsetminus A) = \varnothing$, that is $U_x \subseteq A$. Therefore $x \in A^\circ$.

(c) Since $A^\circ \subseteq A$ and $A^\circ \cap \partial A = \varnothing$ we have $A^\circ \subseteq A \smallsetminus \partial A$.

From parts (a) and (b) we see that $\overline{A}$ is the disjoint union of A° and ∂A; in addition $A \subseteq \overline{A}$ so

$$A \smallsetminus \partial A \subseteq \overline{A} \smallsetminus \partial A = A^\circ.$$ □

2.7. Let $f : X \longrightarrow Y$ be a function and $\mathcal{T}_X$ a topology on X. Define

$$\mathcal{T}_Y = \{U \in \mathcal{P}(Y) : f^{-1}(U) \in \mathcal{T}_X\}.$$

(a) Prove that $\mathcal{T}_Y$ is the finest topology on Y such that f is continuous. (This topology is called the *final topology* induced by f.)

(b) Let $\mathcal{T}$ be another topology on Y. Prove that $f\colon (X, \mathcal{T}_X) \longrightarrow (Y, \mathcal{T})$ is continuous if and only if $\mathcal{T}$ is coarser than $\mathcal{T}_Y$.

Note: There is a "dual" setting where you start with a topology on Y and look for the coarsest topology on Y such that f is continuous, see Exercise 1.23.

Solution.

(a) We start with proving that $\mathcal{T}_Y$ is a topology:

- Since $\varnothing = f^{-1}(\varnothing)$ and $X = f^{-1}(Y)$, it follows that $\mathcal{T}_Y$ contains $\varnothing$ and Y.
- If $\{U_i : i \in I\}$ is a collection of members of $\mathcal{T}_Y$, then
$$\bigcup_{i\in I} f^{-1}(U_i) = f^{-1}\Big(\bigcup_{i\in I} U_i\Big) \in \mathcal{T}_X.$$
- If $U_1, \dots, U_n$ are members of $\mathcal{T}_Y$, then
$$\bigcap_{i=1}^{n} f^{-1}(U_i) = f^{-1}\Big(\bigcap_{i=1}^{n} U_i\Big) \in \mathcal{T}_X.$$

If $\mathcal{T}$ is a topology on Y such that f is continuous, then $f^{-1}(U) \in \mathcal{T}_X$ for every member U of $\mathcal{T}$, so $\mathcal{T} \subseteq \mathcal{T}_Y$. Therefore, $\mathcal{T}_Y$ is the finest topology such that f is continuous.

(b) The 'only if' part has been proven in part (a), so it suffices to prove the 'if' part.

Suppose $\mathcal{T}$ is coarser than $\mathcal{T}_Y$. If U is a member of $\mathcal{T}$, then $U \in \mathcal{T}_Y$, which implies that $f^{-1}(U)$ is open in X. It follows that f is continuous when the topology on Y is $\mathcal{T}$. □

2.8. Prove that a function $f\colon X \longrightarrow Y$ between metric spaces is continuous if and only if it satisfies the usual ε–δ definition: for every point x of X and every positive real number ε, there exists a positive real number δ such that $d_X(x,y) < \delta$ implies $d_Y\big(f(x), f(x)\big) < \varepsilon$.

Solution. It follows from the definition of open balls that the condition '$d_X(x,y) < \delta$ implies $d_Y\big(f(x), f(x)\big) < \varepsilon$' means $f\big(\mathbf{B}_\delta(x)\big) \subseteq \mathbf{B}_\varepsilon\big(f(x)\big)$. We will use the rephrased statement in this proof.

Suppose $f\colon X \longrightarrow Y$ is continuous. If $x \in X$ and ε is a positive real number, then the inverse image of $\mathbf{B}_\varepsilon\big(f(x)\big)$ in X is open, and thus contains $\mathbf{B}_\delta(x)$ for some positive real number δ. It follows that $f\big(\mathbf{B}_\delta(x)\big) \subseteq \mathbf{B}_\varepsilon\big(f(x)\big)$.

Conversely, suppose f satisfies the usual ε–δ definition and consider an open subset U of Y. If $f(x) \in U$ for some element x of X, then the openness of U implies the existence of positive real number ε such that $\mathbf{B}_\varepsilon\big(f(x)\big) \subseteq U$. Since f satisfies the usual ε–δ definition, there exists a positive real number δ such that $f\big(\mathbf{B}_\delta(x)\big) \subseteq \mathbf{B}_\varepsilon\big(f(x)\big) \subseteq U$, which implies $\mathbf{B}_\delta(x) \subseteq f^{-1}(U)$. It follows that $f^{-1}(U)$ is open in X, and thus f is continuous. □

2.9.

(a) Let $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$ be functions, where X, Y, Z are sets, and let $S \subseteq Z$. Then
$$f^{-1}\big(g^{-1}(S)\big) = (g \circ f)^{-1}(S).$$

(b) Let $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$ be continuous functions, where X, Y, Z are topological spaces. Prove that $g \circ f\colon X \longrightarrow Z$ is continuous.

Solution.

(a) We have $x \in (g \circ f)^{-1}(S)$ iff $(g \circ f)(x) \in S$ iff $g(f(x)) \in S$ iff $f(x) \in g^{-1}(S)$ iff $x \in f^{-1}\big(g^{-1}(S)\big)$.

(b) Let $W \subseteq Z$ be open. As $g\colon Y \longrightarrow Z$ is continuous, $g^{-1}(W) \subseteq Y$ is open. As $f\colon X \longrightarrow Y$ is continuous, $(g \circ f)^{-1}(W) = f^{-1}\big(g^{-1}(W)\big) \subseteq X$ is open. So $g \circ f$ is continuous. □