Tutorial Week 3

Topics: Closure, interior, denseness, product, Hausdorff, equivalent metrics, disconnectedness

3.1. Let A and B be subsets of a topological space X.

(a) Suppose $A \subseteq B$. Prove that $\overline{A} \subseteq \overline{B}$ and $A^\circ \subseteq B^\circ$.

(b) Prove that $\overline{A \cup B} = \overline{A} \cup \overline{B}$ and $(A \cap B)^\circ = A^\circ \cap B^\circ$.

(c) Prove that $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$. Find an example in which $\overline{A \cap B} \neq \overline{A} \cap \overline{B}$.

(d) Prove that $(A \cup B)^\circ \supseteq A^\circ \cup B^\circ$. Find an example in which $(A \cup B)^\circ \neq A^\circ \cup B^\circ$.

[**Hint**: For (c) and (d), think of some subsets of $\mathbf{R}$.]

Solution.

(a) Since $A \subseteq B \subseteq \overline{B}$ and $\overline{A}$ is the smallest closed subset of X containing A, it follows that $\overline{A} \subseteq \overline{B}$.

Since $A^\circ \subseteq A \subseteq B$ and B° is the largest open subset of B, it follows that $A^\circ \subseteq B^\circ$.

(b) Since $A \cup B \subseteq \overline{A} \cup \overline{B}$, it follows that $\overline{A \cup B} \subseteq \overline{A} \cup \overline{B}$.

For the other inclusion, $A \subseteq A \cup B \subseteq \overline{A \cup B}$ implies $\overline{A} \subseteq \overline{A \cup B}$, and similarly we have $\overline{B} \subseteq \overline{A \cup B}$. Hence $\overline{A} \cup \overline{B} \subseteq \overline{A \cup B}$.

Since $A^\circ \cap B^\circ \subseteq A \cap B$, it follows that $A^\circ \cap B^\circ \subseteq (A \cap B)^\circ$.

For the other inclusion, $(A \cap B)^\circ \subseteq A \cap B \subseteq A$ implies $(A \cap B)^\circ \subseteq A^\circ$, and similarly we have $(A \cap B)^\circ \subseteq B^\circ$. Hence $(A \cap B)^\circ \subseteq A^\circ \cap B^\circ$.

(c) Both $\overline{A}$ and $\overline{B}$ contain $A \cap B$, so $\overline{A \cap B} \subseteq \overline{A} \cap \overline{B}$.

For the example, let $A = \mathbf{Q}$, $B = \mathbf{R} \setminus \mathbf{Q}$, and $X = \mathbf{R}$. It follows that $\overline{A} = \overline{B} = X$ (see Example 2.23), so $X = \overline{A} \cap \overline{B}$, but $\overline{A \cap B} = A \cap B = \varnothing$.

(d) Since $A \cup B$ contains both A and B, it follows from part (a) that $(A \cup B)^\circ \supseteq A^\circ$ and $(A \cup B)^\circ \supseteq B^\circ$, and hence $(A \cup B)^\circ \supseteq A^\circ \cup B^\circ$.

For the example, let $A = \mathbf{Q}$, $B = \mathbf{R} \setminus \mathbf{Q}$, and $X = \mathbf{R}$. It follows that $(A \cup B)^\circ = X^\circ = X$, but $A^\circ \cup B^\circ = \varnothing$ because $A^\circ = B^\circ = \varnothing$. □

3.2. A subset $D \subseteq X$ of a topological space X is dense in X if and only if $D \cap U \neq \varnothing$ for all nonempty open sets U in X.

Solution. Suppose D is dense, so $\overline{D} = X$, and let U be nonempty open. If $D \cap U = \varnothing$ then $D \subseteq X \setminus U$. But $X \setminus U$ is a closed subset of X containing D, so by the minimality property of $\overline{D}$ we have $\overline{D} \subseteq X \setminus U$. As $U \neq \varnothing$, this means $\overline{D} \neq X$, contradiction.

Conversely, suppose $D \cap U$ is nonempty for any nonempty open U. If $\overline{D} \neq X$ then $U := X \setminus \overline{D}$ is a nonempty open subset of X, so $D \cap (X \setminus \overline{D}) \neq \varnothing$. But this is absurd since $D \subseteq \overline{D}$. □

3.3. Let X be a topological space. The intersection of two dense open sets U_1 and U_2 is dense and open.

Solution. Let $U_{12} = U_1 \cap U_2$. We know already that U_{12} is open.

To show that U_{12} is dense, we use Tutorial Question 3.2 and show that $U_{12} \cap U \neq \varnothing$ for all nonempty open U:

$$U_{12} \cap U = \big(U_1 \cap U_2\big) \cap U = \mathrm{U}_1 \cap \big(U_2 \cap U\big).$$

Since U_2 is dense and open, $U_2 \cap U$ is nonempty and open. Since U_1 is dense, $U_1 \cap \big(U_2 \cap U\big)$ is nonempty. So $U_{12} \cap U \neq \varnothing$, hence U_{12} is dense. □

3.4. Let (X, d) be a metric space and let $A \subseteq X$.

(a) Prove that the set A is open if and only if it is the union of a collection of open balls.

(b) Conclude that the set of all open balls in X generates the metric topology of X.

Solution. (a) In one direction, if A is a union of a collection of open balls, then A is open by Example 2.8 and Proposition 2.9.

In the other direction, suppose A is open. Let $a \in A$, then there exists an open ball $\mathbf{B}_{r(a)}(a) \subseteq A$. Then

$$A = \bigcup_{a \in A} \mathbf{B}_{r(a)}(a).$$

(b) Follows immediately from the definition of the topology generated by a set. □

3.5. Let (X, d) be a metric space.

(a) Prove that the metric topology on (X, d) is generated by open balls of radii smaller than 1.

(b) Define $d' \colon X \times X \longrightarrow \mathbf{R}_{\geqslant 0}$ by

$$d'(x, y) = \min\big\{d(x, y), 1\big\}.$$

Prove that d' is a metric.

(c) Prove that d and d' are equivalent (that is, they give rise to the same topology on X).

Solution.

(a) Since the metric topology on (X, d) is generated by open balls of arbitrary radii, it suffices to prove that every open ball is in the topology generated by open balls of radii smaller than 1. Let x be a point in X and let r be an arbitrary positive real number. If $y \in \mathbf{B}_r(x)$, then $d(x, y) < r$, so $\mathbf{B}_{r-d(x,y)}(y) \subseteq \mathbf{B}_r(x)$ by the triangle inequality. It follows that

$$\mathbf{B}_r(x) = \bigcup_{y \in \mathbf{B}_r(x)} \mathbf{B}_{r-d(x,y)}(y) = \bigcup_{y \in \mathbf{B}_r(x)} \mathbf{B}_{r(y)}(y),$$

where $r(y) = \min\{r - d(x, y), 1\}$. Hence $\mathbf{B}_r(x)$ is in the topology generated by open balls of radii smaller than 1.

(b) It is clear that $d'(y, x) = d'(x, y)$ and that $d'(x, y) = 0$ if and only if $d(x, y) = 0$ if and only if $x = y$.

For the triangle inequality: $d'(x, y) \leqslant 1$ so if at least one of $d'(x, t)$, $d'(t, y)$ is 1, the triangle inequality holds. So we may assume that $d'(x, t) = d(x, t)$ and $d'(t, y) = d(t, y)$. Then

$$d'(x, y) \leqslant d(x, y) \leqslant d(x, t) + d(t, y) = d'(x, t) + d'(t, y).$$

(c) It follows from the definition of d that $\mathbf{B}_r^d(x) = \mathbf{B}_r^{d'}(x)$ for every point x in X and every positive real number r smaller than 1. It then follows from part (a) that the metric topologies induced by d and d' are generated by the same collection of open balls, so the two metric topologies are the same. Hence d and d' are equivalent. □

3.6. Let C_1 and C_2 be two connected subsets of a topological space X such that $C_1 \cap C_2 \neq \varnothing$. Prove that $C_1 \cup C_2$ is connected.

Solution. Let $f\colon C_1 \cup C_2 \longrightarrow \{0,1\}$ be a continuous function, where $\{0,1\}$ is given the discrete topology. Since $C_1 \cap C_2$ is non-empty, we can pick an element x of $C_1 \cap C_2$. By Proposition 2.31, the restrictions of f to C_1 and C_2 are both constant. Hence we have $f(x) = f(y)$ for every element y of $C_1 \cup C_2$; in other words, f is a constant function on $C_1 \cup C_2$. By Proposition 2.31, this implies $C_1 \cup C_2$ is connected. □

3.7. Let X be a topological space and define $x \sim x'$ if there exists a connected subset $C \subseteq X$ such that $x, x' \in C$. Prove that this is an equivalence relation on the set X.

(The equivalence classes are called the *connected components* of X).

Solution.

(a) $x \sim x$: for any $x \in X$, the set $C = \{x\}$ is connected and contains x, so $x \sim x$.

(b) if $x \sim x'$ then $x' \sim x$: clear from the definition, which does not distinguish x and x'.

(c) if $x \sim x'$ and $x' \sim x''$ then $x \sim x''$: since $x \sim x'$ there exists a connected set C_1 such that $x, x' \in C_1$; since $x' \sim x''$ there exists a connected set C_2 such that $x', x'' \in C_2$; by Tutorial Question 3.6, since C_1 and C_2 are connected and $x' \in C_1 \cap C_2$, the union $C_1 \cup C_2$ is connected, and it contains both x and x'', so that $x \sim x''$. □

3.8. Let X, Y_1, and Y_2 be topological spaces, and $\pi_1\colon Y_1 \times Y_2 \longrightarrow Y_1$ and $\pi_2\colon Y_1 \times Y_2 \longrightarrow Y_2$ be the projections. Prove that a function $f\colon X \longrightarrow Y_1 \times Y_2$ is continuous if and only if both $\pi_1 \circ f$ and $\pi_2 \circ f$ are continuous.

Solution. Suppose f is continuous. By Proposition 2.18, both π_1 and π_2 are continuous. It then follows from Tutorial Question 2.9 that $\pi_1 \circ f$ and $\pi_2 \circ f$ are continuous.

Conversely, suppose $\pi_1 \circ f$ and $\pi_2 \circ f$ are both continuous. If U_1 and U_2 are open subsets of Y_1 and Y_2 respectively, then $f^{-1}(U_1 \times U_2) = (\pi_1 \circ f)^{-1}(U_1) \cap (\pi_2 \circ f)^{-1}(U_2)$ because

$$\begin{aligned}
x \in f^{-1}(U_1 \times U_2) &\iff \big((\pi_1 \circ f)(x), (\pi_2 \circ f)(x)\big) \in U_1 \times U_2 \\
&\iff (\pi_1 \circ f)(x) \in U_1 \text{ and } (\pi_2 \circ f)(x) \in U_2 \\
&\iff x \in (\pi_1 \circ f)^{-1}(U_1) \text{ and } x \in (\pi_2 \circ f)^{-1}(U_2) \\
&\iff x \in (\pi_1 \circ f)^{-1}(U_1) \cap (\pi_2 \circ f)^{-1}(U_2).
\end{aligned}$$

Since $\pi_1 \circ f$ and $\pi_2 \circ f$ are both continuous, it follows that $(\pi_1 \circ f)^{-1}(U_1)$ and $(\pi_2 \circ f)^{-1}(U_2)$ are both open in X; hence $f^{-1}(U_1 \times U_2)$ is open. The topology on $Y_1 \times Y_2$ is generated by rectangles, so it follows from Exercise 1.22 that f is continuous. □

3.9. Given a set X, define the *diagonal function*

$$\Delta\colon X \longrightarrow X \times X, \qquad x \longmapsto (x, x).$$

(a) Prove that two subsets A and B of X are disjoint if and only if $\Delta(X)$ and $A \times B$ are disjoint.

(b) If X is a topological space, prove that Δ is continuous.

(c) Prove that a topological space X is Hausdorff if and only if $\Delta(X)$ is closed in $X \times X$.

Solution. (a) This follows from

$$x \in A \cap B \iff x \in A \text{ and } x \in B \iff (x,x) \in \Delta(X) \cap (A \times B).$$

(b) Let $\mathrm{id}_X : X \longrightarrow X$ denote the identity function of X, defined by $\mathrm{id}_X(x) = x$. This function is continuous by Exercise 1.21.

Let $\pi_1, \pi_2 : X \times X \longrightarrow X$ be the projections. Since $\pi_1 \circ \Delta = \pi_2 \circ \Delta = \mathrm{id}_X$, it follows from Tutorial Question 3.8 that Δ is continuous.

(c) Suppose X is Hausdorff. We will prove that $(X \times X) \smallsetminus \Delta(X)$ is open. If $(x,y) \in (X \times X) \smallsetminus \Delta(X)$, then $x \neq y$, so there exist disjoint open neighbourhoods U of x and V of y. It follows from part (a) that $U \times V$ does not intersect $\Delta(X)$, and is thus a subset of $(X \times X) \smallsetminus \Delta(X)$. This implies $(X \times X) \smallsetminus \Delta(X)$ is open, and hence $\Delta(X)$ is closed.

Conversely, suppose $\Delta(X)$ is closed; in other words, $(X \times X) \smallsetminus \Delta(X)$ is open. Let x and y be two distinct points in X. Since $(x,y) \in (X \times X) \smallsetminus \Delta(X)$, it follows that there exist open neighbourhoods U of x and V of y such that $U \times V \subseteq (X \times X) \smallsetminus \Delta(X)$, which implies U and V are disjoint by part (a). Hence X is Hausdorff. □