Tutorial Week 4

Topics: Closed functions, compactness, sequences

4.1. Let X, Y be topological spaces. Recall that a function $f\colon X \longrightarrow Y$ is said to be *closed* if for every closed subset $C \subseteq X$, the image $f(C)$ is closed in Y.

(a) Prove that the composition of two closed maps is a closed map.

(b) Prove that a continuous bijection between topological spaces is a homeomorphism if and only if it is closed.

(c) Give an example of a bijection $f\colon X \longrightarrow Y$ between topological spaces such that f is continuous but not closed (and therefore f^{-1} is closed but not continuous).

Solution.

(a) Let $f\colon X \longrightarrow Y$ and $g\colon Y \longrightarrow Z$ be the two closed maps. Suppose F is a closed subset of X. Since f is a closed map, it follows that $f(F)$ is a closed subset. Since g is a closed map, we see that $g(f(F)) = (g \circ f)(F)$ is a closed subset. Hence $g \circ f$ is a closed map.

(b) Let $f\colon X \longrightarrow Y$ be a continuous bijection between topological spaces and let $g\colon Y \longrightarrow X$ be its inverse.

Suppose f is a homeomorphism. If F is a closed subset of X, then $f(F) = g^{-1}(F)$ is closed by continuity of g. Hence f is a closed map.

Conversely, suppose f is a closed map. If F is a closed subset of X, then $g^{-1}(F) = f(F)$ is closed. It then follows from Exercise 1.17 that g is continuous, and therefore f is a homeomorphism.

(c) Let $X = \{0, 1\}$ with the discrete topology, let $Y = \{0, 1\}$ with the trivial (indiscrete) topology, and let f be the identity map. The claims follow easily. □

4.2. Let X, Y be topological spaces. Recall that a function $f\colon X \longrightarrow Y$ is said to be *open* if for every open set $U \subseteq X$, the image $f(U)$ is open in Y.

(a) Give an example of a function that is open but not closed.

(b) Give an example of a function that is closed but not open.

Solution.

(a) Let $X = \{0, 1\}$ with the discrete topology, and $Y = \{0, 1\}$ with the topology generated by $\{0\}$. Then as a subset of Y, $\{0\}$ is open but not closed.

Therefore the function $f\colon X \longrightarrow Y$ with $f(0) = f(1) = 0$ is open but not closed.

(b) With the same X and Y as in part (a), note that as a subset of Y, $\{1\}$ is closed but not open.

Therefore the function $g\colon X \longrightarrow Y$ with $g(0) = g(1) = 1$ is open but not closed. □

4.3. Let K and L be compact subsets of a topological space X. Prove that $K \cup L$ is compact.

Solution. Consider an arbitrary open cover of $K \cup L$:

$$K \cup L \subseteq \bigcup_{i \in I} U_i.$$

This is also an open cover of K, so there is a finite subcover that still covers K:

$$K \subseteq \bigcup_{n=1}^{N} U_{i_n}, \qquad i_n \in I.$$

Similarly, we get a finite subcover that covers L:

$$L \subseteq \bigcup_{m=1}^{M} U_{j_m}, \qquad j_m \in I.$$

Letting $S = \{i_1, \dots, i_N\} \cup \{j_1, \dots, j_M\}$, we get a finite subcover that covers $K \cup L$:

$$K \cup L \subseteq \bigcup_{s \in S} U_s. \qquad \square$$

4.4. Prove that every finite topological space is compact.

Solution. Let X be a finite topological space and consider an open cover $\{U_i : i \in I\}$ of X. For every point x in X, pick a member U_x of $\{U_i : i \in I\}$ such that $x \in U_x$. Now $\{U_x : x \in X\}$ is a finite sub-cover of $\{U_i : i \in I\}$. Hence X is compact. $\square$

4.5. Prove that a discrete topological space X is compact if and only if X is finite.

Solution. If X is finite, then X is compact by Tutorial Question 4.4.

Conversely, suppose X is compact and consider the open cover $\big\{\{x\} : x \in X\big\}$. Its only subcover is itself (any proper subcollection will miss some points of X), but by compactness it admits a finite subcover, so the cover itself must have been finite, hence X is finite. $\square$

4.6. Let X be a compact topological space and let Y be a Hausdorff topological space. Prove that every continuous bijection from X to Y is a homeomorphism.

Solution. Let $f : X \longrightarrow Y$ be a continuous bijection. We will prove f is a closed map; it will then follow from part (b) of Tutorial Question 4.1 that f is a homeomorphism.

If F is a closed subset of X, then it is compact by Proposition 2.38. It follows from Proposition 2.39 that $f(F)$ is compact, which implies it is a closed subset by Proposition 2.37. Hence f is a closed map. $\square$

4.7. Let (x_n) be a sequence in a metric space X, let $\varphi : \mathbf{N} \longrightarrow \mathbf{N}$ be an injective function, and consider the sequence $(y_n) = \big(x_{\varphi(n)}\big)$ in X. Prove that if (x_n) converges to x, then so does (y_n).

Does the converse hold?

Solution. Suppose $(x_n) \longrightarrow x$. Given $\varepsilon > 0$, let $N \in \mathbf{N}$ be such that $x_n \in \mathbf{B}_\varepsilon(x)$ for all $n \geqslant N$.

Since $\varphi : \mathbf{N} \longrightarrow \mathbf{N}$ is injective, the inverse image $\varphi^{-1}\big(\{1, \dots, N-1\}\big)$ is a finite set, so it has a maximal element M. (If the set is empty, just take $M = 0$.) For all $n \geqslant M + 1$, we have $\varphi(n) \geqslant N$, so $y_n = x_{\varphi(n)} \in \mathbf{B}_\varepsilon(x)$.

The converse certainly does not hold. For instance, take $(x_n) = (1, 0, 1, 0, 1, 0, \dots)$ and $\varphi(n) = 2n$, then the sequence $(y_n) = (0, 0, 0, \dots)$ converges to 0 but (x_n) does not converge. $\square$

4.8. Give $\mathbf{N} \subseteq \mathbf{R}$ the subspace topology. Let X be a topological space and (x_n) a sequence in X. Prove that (x_n) is a continuous function $\mathbf{N} \longrightarrow X$.

Solution. First note that the subspace topology on $\mathbf{N} \subseteq \mathbf{R}$ is the discrete topology: for any $n \in \mathbf{N}$, we have $\{n\} = (n-1, n+1) \cap \mathbf{N}$, so $\{n\}$ is open in $\mathbf{N}$. Therefore every subset of $\mathbf{N}$ is open, hence every function $\mathbf{N} \longrightarrow X$ is continuous. □

4.9. Let (X, d_X) and (Y, d_Y) be metric spaces and let d be the sup norm metric on $X \times Y$:

$$d\big((x_1, y_1), (x_2, y_2)\big) = \max\big(d_X(x_1, x_2), d_Y(y_1, y_2)\big).$$

Prove that $\big((x_n, y_n)\big) \longrightarrow (x, y) \in X \times Y$ if and only if $(x_n) \longrightarrow x \in X$ and $(y_n) \longrightarrow y \in Y$.

Solution. Suppose $(x_n) \longrightarrow x$ and $(y_n) \longrightarrow y$. Let $\varepsilon > 0$, $N_x \in \mathbf{N}$ such that $x_n \in \mathbf{B}_\varepsilon(x)$ for all $n \geqslant N_x$, and $N_y \in \mathbf{N}$ such that $y_n \in \mathbf{B}_\varepsilon(y)$ for all $n \geqslant N_y$. Set $N = \max\{N_x, N_y\}$, then

$$d\big((x_n, y_n), (x, y)\big) = \max\{d_X(x_n, x), d_Y(y_n, y)\} < \varepsilon \quad \text{for all } n \geqslant N.$$

Conversely, suppose $\big((x_n, y_n)\big) \longrightarrow (x, y)$. Given $\varepsilon > 0$ there exists $N \in \mathbf{N}$ such that $(x_n, y_n) \in \mathbf{B}_\varepsilon\big((x, y)\big)$ for all $n \geqslant N$, so

$$\max\{d_X(x_n, x), d_Y(y_n, y)\} = d\big((x_n, y_n), (x, y)\big) < \varepsilon,$$

and hence both $d_X(x_n, x)$ and $d_Y(y_n, y)$ are bounded by ε for all $n \geqslant N$.

(Alternative solution): Define a function $f\colon \mathbf{N}^* \longrightarrow X \times Y$ by

$$f(n) = \begin{cases} (x_n, y_n) & \text{if } n \in \mathbf{N}, \\ (x, y) & \text{otherwise.} \end{cases}$$

Let $\pi_X\colon X \times Y \longrightarrow X$ and $\pi_Y\colon X \times Y \longrightarrow Y$ be the projections. The result follows from the following:

- f is continuous if and only if π_X and π_Y are both continuous (see Tutorial Question 3.8).
- f is continuous if and only if (x_n, y_n) converges to (x, y) (part (c) of Exercise 1.56).
- $\pi_X \circ f$ is continuous if and only if x_n converges to x (part (c) of Exercise 1.56).
- $\pi_Y \circ f$ is continuous if and only if y_n converges to y (part (c) of Exercise 1.56). □