Tutorial Week 5

Topics: Sequences, completeness, uniform continuity

5.1. Let d_1 and d_2 be equivalent metrics (they define the same topology) on a set X. Prove that a sequence converges to a point x in (X, d_1) if and only if it converges to x in (X, d_2).

Solution. Since d_1 and d_2 are interchangeable, it suffices to prove the 'only if' part. Let (x_n) be a sequence converging to x in (X, d_1). By Proposition 2.15, the identity function $\mathrm{id}_X : X \longrightarrow X$ defined by $\mathrm{id}_X(x) = x$ is continuous as a function from (X, d_1) to (X, d_2). The result then follows from Theorem 2.44. □

5.2. Let $f : X \longrightarrow Y$ and $g : Y \longrightarrow Z$ be uniformly continuous functions between metric spaces. Prove that $g \circ f : X \longrightarrow Z$ is uniformly continuous.

Solution. Let ε be a positive real number. The uniform continuity of f and g implies that there exists a positive real number δ such that $d_Y(y_1, y_2) < \delta$ implies $d_Z\big(g(y_1), g(y_2)\big) < \varepsilon$, and there exists a positive real number γ such that $d_X(x_1, x_2) < \gamma$ implies $d_Y\big(f(x_1), f(x_2)\big) < \delta$. Hence $d_X(x_1, x_2) < \gamma$ implies $d_Z\big((g \circ f)(x_1), (g \circ f)(x_2)\big) < \varepsilon$, and therefore $g \circ f$ is uniformly continuous. □

5.3. Let $f : X \longrightarrow Y$ be a uniformly continuous function between two metric spaces and suppose $(x_n) \sim (x'_n)$ are equivalent sequences in X. Prove that $\big(f(x_n)\big) \sim \big(f(x'_n)\big)$ as sequences in Y.

Does the conclusion hold if f is only assumed to be continuous?

Solution. Let $\varepsilon > 0$. As f is uniformly continuous, there exists $\delta > 0$ such that for all $x, x' \in X$, if $d_X(x, x') < \delta$ then $d_Y(f(x), f(x')) < \varepsilon$. As $(x_n) \sim (x'_n)$, there exists $N \in \mathbf{N}$ such that $d_X(x_n, x'_n) < \delta$ for all $n \geqslant N$. Hence for all $n \geqslant N$ we have $d_Y(f(x_n), f(x'_n)) < \varepsilon$.

The result does not hold in general for continuous functions; for instance one can take $f : \mathbf{R}_{>0} \longrightarrow \mathbf{R}_{>0}$ given by $f(x) = \frac{1}{x}$, and $(1/n) \sim (1/n^2)$ but $(f(1/n)) = (n)$, $(f(1/n^2)) = (n^2)$ and $(n) \not\sim (n^2)$. □

5.4. Let X be a complete metric space and let $S \subseteq X$. Prove that the closure $\overline{S}$ (with the metric induced from $\overline{S} \subseteq X$) is a completion of S (with the metric induced from $S \subseteq X$).

Solution. Of course, $\overline{S}$ is complete: if (x_n) is a Cauchy sequence in $\overline{S}$, then it is a Cauchy sequence in X, so $(x_n) \longrightarrow x \in X$ since X is complete. But $\overline{S}$ is closed, so $(x_n) \longrightarrow x \in \overline{S}$.

We let $\iota : S \longrightarrow \overline{S}$ be the inclusion map: $\iota(s) = s$ for all $s \in S$. It is injective and an isometry (as d_S and $d_{\overline{S}}$ are both induced from d_X).

Finally, S is dense in $\overline{S}$: by Proposition 2.42, for every $x \in \overline{S}$ there exists a sequence (s_n) in S such that $(s_n) \longrightarrow x$. □

5.5. Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \longrightarrow Y$ a surjective continuous function. Suppose that X is complete and for all $x_1, x_2 \in X$ we have

$$d_X\big(x_1, x_2\big) \leqslant d_Y\big(f(x_1), f(x_2)\big).$$

Prove that Y is complete.

In particular, isometries preserve completeness.

Solution. Let (y_n) be a Cauchy sequence in Y. For each $n \in \mathbf{N}$, let $x_n \in f^{-1}(y_n)$. I claim that (x_n) is a Cauchy sequence in X. Fix $\varepsilon > 0$. Let $N \in \mathbf{N}$ be such that for all $m, n \geqslant N$ we have $d_Y(y_m, y_n) < \varepsilon$. Then for all $m, n \geqslant N$ we have

$$d_X(x_m, x_n) \leqslant d_Y\big(f(x_m), f(x_n)\big) = d_Y(y_m, y_n) < \varepsilon,$$

so (x_n) is indeed Cauchy in X.

Since X is complete, we have $(x_n) \longrightarrow x \in X$, so that by the continuity of f we conclude that $(y_n) = \big(f(x_n)\big) \longrightarrow f(x) \in Y$. □

5.6. Let (X, d) be a metric space and let $S \subseteq X$ be a nonempty subset. Define $d_S\colon X \longrightarrow \mathbf{R}_{\geqslant 0}$ by

$$d_S(x) = \inf_{s \in S} d(x, s).$$

(a) Prove that d_S is uniformly continuous.

[**Hint**: Show that $|d_S(x) - d_S(y)| \leqslant d(x, y)$ for all $x, y \in X$.]

(b) Prove that $d_S(x) = 0$ if and only if $x \in \overline{S}$.

(c) Prove that if $U \subseteq X$ is an open neighbourhood of x, then $d_{X \setminus U}(x) > 0$.

Solution.

(a) We start with the hint. Let $x, y \in X$. For all $s \in S$ we have

$$d_S(x) \leqslant d(x, s) \leqslant d(x, y) + d(y, s),$$

hence

$$d_S(x) \leqslant d(x, y) + d_S(y).$$

We can swap the roles of x and y to get

$$d_S(y) \leqslant d(y, x) + d_S(x),$$

and the two inequalities together give

$$|d_S(x) - d_S(y)| \leqslant d(x, y).$$

Uniform continuity is now clear: for any $\varepsilon > 0$ we take $\delta = \varepsilon$ and use the above inequality.

(b) If $d_S(x) = 0$ then $\inf d(x, s) = 0$ so for any $\varepsilon > 0$ there exists $s \in S$ such that $d(x, s) < \varepsilon$. In particular, for $n \in \mathbf{N}$ we can set $\varepsilon = 1/n$ and get $s_n \in S$ such that $d(x, s_n) < \varepsilon$. This gives us a sequence (s_n) in S that converges to x, so $x \in \overline{S}$.

Conversely, if $x \in \overline{S}$ then there exists a sequence (s_n) in S that converges to x. Given $\varepsilon > 0$, there exists $N \in \mathbf{N}$ such that $d(x, s_N) < \varepsilon$, therefore $\inf d(x, s) = 0$.

(c) If $d_{X \setminus U}(x) = 0$ then by part (b) we have $x \in \overline{X \setminus U} = X \setminus U$, the latter equality due to U being open. But then $x \in U \cap (X \setminus U)$, contradiction. □