Tutorial Week 6

Topics: Contractions, Banach fixed point theorem

6.1. Prove that any contraction is uniformly continuous.

Solution. Suppose $f\colon X \longrightarrow Y$ is a contraction with constant C.
Let $\varepsilon > 0$ and set $\delta = \frac{\varepsilon}{C+1}$, then for all $x_1, x_2 \in X$ such that $d_X(x_1, x_2) < \delta$, we have

$$d_Y(f(x_1), f(x_2)) \leqslant C d_X(x_1, x_2) \leqslant C\delta = \frac{C}{C+1}\varepsilon < \varepsilon. \qquad \square$$

6.2. Consider the equation

$$x^3 - x - 1 = 0. \tag{1}$$

(a) Show that the equation must have **at least one solution** in the interval $[1, 2]$.

(b) Show that the function $f\colon [1, 2] \longrightarrow \mathbf{R}$ given by

$$f(x) = (1 + x)^{1/3}$$

has image contained in $[1, 2]$ and is a contraction.

(c) Show that Equation (1) has **a unique solution** ξ in the interval $[1, 2]$ and describe a sequence of real numbers that converges to ξ.

Solution.

(a) We can use the Intermediate Value Theorem: at $x = 1$, $x^3 - x - 1 = -1 < 0$, while at $x = 2$, $x^3 - x - 1 = 5 > 0$, so there must be at least one point x in $[1, 2]$ such that $x^3 - x - 1 = 0$.

(b) Given $x \in [1, 2]$ we have

$$1 \leqslant x \leqslant 2 \quad \Rightarrow \quad 2 \leqslant 1 + x \leqslant 3 \quad \Rightarrow \quad 1 \leqslant 2^{1/3} \leqslant (1+x)^{1/3} \leqslant 3^{1/3} \leqslant 2,$$

since $t \longmapsto t^{1/3}$ is an increasing function.

The derivative of f is

$$f'(x) = \frac{1}{3}(1+x)^{-2/3} = \frac{1}{3}\frac{1}{(1+x)^{2/3}}.$$

We note that $f'(x) > 0$ on $[1, 2]$ and

$$1 \leqslant x \Rightarrow 2 \leqslant 1 + x \Rightarrow \frac{1}{1+x} \leqslant \frac{1}{2} \Rightarrow \frac{1}{(1+x)^{2/3}} \leqslant \frac{1}{2^{2/3}} \leqslant 1,$$

so that

$$f'(x) \leqslant \frac{1}{3}.$$

Now let x, y be such that $1 \leqslant x < y \leqslant 2$ and apply the Mean Value Theorem to f on $[x, y]$ to deduce that there exists $c \in (x, y)$ such that

$$\frac{f(y) - f(x)}{y - x} = f'(c) \Rightarrow |f(y) - f(x)| = |f'(c)|\,|y - x| \leqslant \frac{1}{3}|y - x|.$$

We conclude that f is a contraction.

(c) Observe that $x^3 - x - 1 = 0$ is equivalent to $f(x) = x$, so the solutions of Equation (1) are precisely the fixed points of f. As f is a contraction and $[1,2]$ is complete, the Banach Fixed Point Theorem says that there is a unique fixed point ξ in $[1,2]$. It also tells us that we can start with any $x_1 \in [1,2]$, for instance $x_1 = 1$, and iteratively apply f to get a sequence (x_n) converging to ξ:

$$x_1 = 1, \qquad x_2 = f(x_1) = 2^{1/3}, \qquad x_3 = f(x_2) = \left(1 + 2^{1/3}\right)^{1/3}, \ldots \qquad \square$$

6.3. Find a non-empty metric space X and a contraction $f\colon X \longrightarrow X$ such that f has no fixed points.

Solution. Let $X = (0, \infty)$, which is given the Euclidean topology, and let $f\colon X \longrightarrow X$ be the function defined by $f(x) = x/2$. This is a contraction because if x and y are positive real numbers then

$$d_X\big(f(x), f(y)\big) = \left|\frac{x-y}{2}\right| = \frac{1}{2}|x-y| = \frac{1}{2}d_X(x,y).$$

It has no fixed points because $f(x) = x$ implies $x = 0$, but $0 \notin (0, \infty)$. $\square$

6.4. Recall Newton's method for solving equations: given a differentiable function g and an initial guess x_0, iterate

$$x_{n+1} = x_n - \frac{g(x_n)}{g'(x_n)}, \qquad n \geqslant 0.$$

The aim is to get a sequence (x_n) that converges to a root of g.

Apply this to the function $g(x) = x^2 - 3$:

(a) Prove that $f(x) := x - g(x)/g'(x)$ defines a contraction from $X = [\sqrt{3}, \infty)$ to itself.

(b) Use the Banach Fixed Point Theorem to conclude that the Newton iteration converges to $\sqrt{3}$ for any starting point $x_0 \in X = [\sqrt{3}, \infty)$.

(c) What happens if we pick a starting point $x_0 \in (0, \sqrt{3})$?

Solution.

(a) We have

$$f(x) = \frac{1}{2}\left(x + \frac{3}{x}\right).$$

The derivative is

$$f'(x) = \frac{1}{2}\left(1 - \frac{3}{x^2}\right),$$

so if $x \in X$ then $x \geqslant \sqrt{3}$ so $x^2 \geqslant 3$ so $f'(x) \geqslant 0$. In other words, f is non-decreasing on X, so the minimal value is attained at $x = \sqrt{3}$: $f(\sqrt{3}) = \sqrt{3}$. Hence f maps X to X.

To see that f is a contraction, let $x, y \in X$:

$$d(f(x), f(y)) = |f(x) - f(y)| = \frac{1}{2}|x-y|\left|1 - \frac{3}{xy}\right| \leqslant \frac{1}{2}|x-y| = \frac{1}{2}d(x,y).$$

(b) By the Banach Fixed Point Theorem, we can take any $x_0 \in X$ and repeatedly apply f (precisely what the Newton's method iteration is), and we get a sequence that converges to the unique fixed point of f. But a fixed point of f gives a root of g in X, which must be $\sqrt{3}$.

(c) If $x_0 \in (0, \sqrt{3})$, then
$$f(x_0) = \frac{1}{2}\left(x + \frac{3}{x}\right) > \sqrt{3}.$$
To check the last inequality, we can apply the reasoning from (a): $f'(x) < 0$ on $(0, \sqrt{3})$ so f is decreasing on $(0, \sqrt{3})$, therefore $f(x_0) > f(\sqrt{3}) = \sqrt{3}$.

At this point we can apply the previous part with starting point $f(x_0) \in X$ and get the same conclusion. □

6.5. Let $A = (a_{ij})$ be an $n \times n$ real matrix with all $|a_{ij}| < 1$.
Given a nonzero real eigenvalue λ of A, consider the function $f_\lambda \colon \mathbf{R}^n \longrightarrow \mathbf{R}^n$ given by
$$f_\lambda(v) = \frac{1}{\lambda} Av.$$

(a) Prove that if $|\lambda| \geqslant n$ then f_λ is a contraction for the sup metric d on $\mathbf{R}^n$:
$$d(x, y) = \max_{i \in \{1,\dots,n\}} |x_i - y_i|, \qquad x = \begin{pmatrix} x_1 & \dots & x_n \end{pmatrix}^\mathsf{T}, y = \begin{pmatrix} y_1 & \dots & y_n \end{pmatrix}^\mathsf{T} \in \mathbf{R}^n.$$

(b) Use the Banach Fixed Point Theorem to derive a contradiction, and thus conclude that every real eigenvalue λ of A satisfies $|\lambda| < n$.

Solution.

(a) Since $|a_{ij}| < 1$ for all $i, j = 1, \dots, n$, letting $C := \max_{i,j} |a_{ij}|$ we have $0 \leqslant C < 1$.

Suppose $|\lambda| \geqslant n$. If v and w are elements of $\mathbf{R}^n$, then
$$\begin{aligned}
d\big(f_\lambda(v), f_\lambda(w)\big) &= \max_{i \in \{1,\dots,n\}} |f_\lambda(v)_i - f_\lambda(w)_i| = \max_i \left| \sum_{j=1}^n \frac{1}{\lambda} a_{ij}(v_j - w_j) \right| \\
&= \frac{1}{|\lambda|} \max_i \left| \sum_{j=1}^n a_{ij}(v_j - w_j) \right| \leqslant \frac{1}{|\lambda|} \max_i \sum_{j=1}^n |a_{ij}| \, |v_j - w_j| \\
&< \frac{\max_i |a_{ij}|}{|\lambda|} \max_i \sum_{j=1}^n |v_j - w_j| = \frac{\max_i |a_{ij}|}{|\lambda|} \sum_{j=1}^n |v_j - w_j| \\
&\leqslant \frac{n \max_i |a_{ij}|}{|\lambda|} \max_{j \in \{1,\dots,n\}} |v_j - w_j| = \frac{n \max_i |a_{ij}|}{|\lambda|} d(v, w) \\
&\leqslant \max_i |a_{ij}| \, d(v, w) \leqslant C \, d(v, w).
\end{aligned}$$

Hence f_λ is a contraction.

(b) Assume there is an eigenvalue satisfying $|\lambda| \geqslant n$. $\mathbf{R}$ is complete with respect to the sup metric (since when $n = 1$ the sup metric is the same as the Euclidean metric on $\mathbf{R}$). Therefore $\mathbf{R}^n$ is complete with respect to the sup metric by Exercise 1.63. Now it follows from the Banach Fixed Point Theorem (Theorem 2.59) that there is a unique element v of $\mathbf{R}^n$ such that $v = f_\lambda(v) = \frac{1}{\lambda} Av$, which is equivalent to $Av = \lambda v$. Since the zero vector satisfies this condition, this unique vector has to be the zero vector, so λ cannot be an eigenvalue of A, contradiction. □