Tutorial Week 7

Topics: (Total) boundedness, uniform convergence

7.1. Prove that in any metric space (X, d), any totally bounded set S is bounded.

Solution. Take $\varepsilon = 1$ and let $B_1, \dots, B_N$ be a cover of S by open balls of radius 1. Each B_n is bounded, so by Exercise 1.73 the finite union $B_1 \cup \dots \cup B_N$ is bounded, hence so is its subset S. □

7.2. Find a bounded subset of a metric space that is not totally bounded.

Solution. Endow $\mathbf{N}$ with the discrete topology. The set $\mathbf{N}$ is bounded because $\mathbf{N} = \mathbf{B}_2(0)$. However, if n is an element of $\mathbf{N}$, then $\mathbf{B}_1(n) = \{n\}$, so it is impossible to cover $\mathbf{N}$ by finitely many open balls of radius 1. □

7.3. Let (X, d) be a metric space.

Prove that if A and B are bounded sets with $A \cap B \neq \varnothing$, then

$$\operatorname{diam}(A \cup B) \leqslant \operatorname{diam}(A) + \operatorname{diam}(B).$$

What happens if $A \cap B = \varnothing$?

Solution. It suffices to show that for any $x, y \in A \cup B$ we have

$$d(x, y) \leqslant \operatorname{diam}(A) + \operatorname{diam}(B).$$

If $x, y \in A$, this is obvious as $d(x, y) \leqslant \operatorname{diam}(A)$. Similarly if $x, y \in B$.

It remains to see what happens if $x \in A$ and $y \in B$. Let $t \in A \cap B$. We have

$$d(x, y) \leqslant d(x, t) + d(t, y) \leqslant \operatorname{diam}(A) + \operatorname{diam}(B),$$

as desired.

If $A \cap B = \varnothing$ we have no control over $\operatorname{diam}(A \cup B)$, as one can see by taking $A = \{x\}$ and $B = \{y\}$. Then $\operatorname{diam}(A \cup B) = d(x, y)$ can be arbitrary, while $\operatorname{diam}(A) + \operatorname{diam}(B) = 0 + 0 = 0$. □

7.4.

(a) Prove that every subspace of a totally bounded space is totally bounded.

(b) Suppose a metric space X has a totally bounded dense subset D. Prove that X is totally bounded.

(c) Prove that a metric space X is totally bounded if and only if it is isometric to a subspace of a compact metric space. [**Hint**: Completion.]

Solution.

(a) Let S be a subspace of a totally bounded space X. If (x_n) be a sequence in S, then it is also a sequence in X, so it has a Cauchy subsequence by Proposition 2.65. Now it again follows from Proposition 2.65 that S is totally bounded.

(b) Let ε be a positive real number. Since D is totally bounded, there exists a natural number N and elements $x_1, \dots, x_N$ of D such that

$$D \subseteq \bigcup_{n=1}^{N} \mathbf{B}_{\varepsilon/2}(x_n).$$

Since X is the closure of D in X, it follows that

$$X \subseteq \bigcup_{n=1}^{N} \overline{\mathbf{B}_{\varepsilon/2}(x_n)} \subseteq \bigcup_{n=1}^{N} \mathbf{B}_{\varepsilon}(x_n).$$

(c) Suppose a metric space X is totally bounded and let $\widehat{X}$ be a completion of X with isometry $\iota\colon X \longrightarrow \widehat{X}$. By the definition of completion, we know that X is isometric to $\iota(X)$, so $\iota(X)$ is totally bounded by Proposition 2.64. It follows from part (b) that the completion $\widehat{X}$ is totally bounded, and is therefore compact by the Heine–Borel theorem (Theorem 2.66). Hence X is isometric to the subspace $\iota(X)$ of the compact metric space $\widehat{X}$.

Conversely, suppose Y is a compact subspace, S is a subspace of Y, and $f\colon S \longrightarrow X$ is a bijective isometry. It follows from the Heine–Borel theorem (Theorem 2.66) that Y is totally bounded, and therefore S is totally bounded by part (a). Hence $X = f(S)$ is totally bounded by Proposition 2.64. □

7.5. We say that a topological space is *separable* if it contains a countable dense subset. (Easy examples are $\mathbf{R}$ with countable dense subset $\mathbf{Q}$, or more generally $\mathbf{R}^n$ with countable dense subset $\mathbf{Q}^n$.)

Prove that any totally bounded metric space X is separable.

Solution. For a fixed $n \in \mathbf{Z}_{\geqslant 1}$, cover X with a finite number of open balls of radius $\frac{1}{n}$ and let $D_n \subseteq X$ be the set of centres of these balls. Now let

$$D = \bigcup_{n=1}^{\infty} D_n.$$

This is a countable union of finite sets, hence countable.

Now take $x \in X$ and $\varepsilon > 0$. Let $n \in \mathbf{N}$ be such that $\frac{1}{n} < \varepsilon$. Since X is covered by the open balls of radius $\frac{1}{n}$ centred at elements of D_n, there exists $y \in D_n \subseteq D$ such that $x \in \mathbf{B}_{1/n}(y)$, that is $d(x, y) < \frac{1}{n} < \varepsilon$. So D is dense in X. □

7.6. Given metric spaces X, Y, prove that a sequence (f_n) in $B(X, Y)$ converges uniformly to $f \in B(X, Y)$ if and only if $(f_n) \longrightarrow f$ with respect to the uniform metric d_∞ on $B(X, Y)$.

Solution. Suppose (f_n) converges uniformly to f. Given $\varepsilon > 0$, there exists $N \in \mathbf{N}$ such that for all $n \geqslant N$ we have

$$d_Y(f_n(x), f(x)) < \frac{\varepsilon}{2} \qquad \text{for all } x \in X.$$

So for all $n \geqslant N$ we have

$$d_\infty(f_n, f) = \sup_{x \in X}\{d_Y(f_n(x), f(x))\} \leqslant \frac{\varepsilon}{2} < \varepsilon,$$

in other words $(f_n) \longrightarrow f$ w.r.t. d_∞.

Conversely, suppose $(f_n) \longrightarrow f$. Given $\varepsilon > 0$, there exists $N \in \mathbf{N}$ such that for all $n \geqslant N$ we have

$$\sup_{x \in X}\{d_Y(f_n(x), f(x))\} = d_\infty(f_n, f) < \varepsilon,$$

hence for all $n \geqslant N$

$$d_Y(f_n(x), f(x)) < \varepsilon \qquad \text{for all } x \in X,$$

in other words (f_n) converges uniformly to f. □

7.7. Give an example of a sequence of bounded continuous functions that converges pointwise to a discontinuous function.

[**Hint**: Consider the behaviour of x^n as $n \longrightarrow \infty$.]

Solution. For $n \in \mathbf{N}$, take $f_n\colon [0,1] \longrightarrow \mathbf{R}$ given by $f_n(x) = x^n$, then the pointwise limit is

$$f\colon [0,1] \longrightarrow \mathbf{R}, \qquad f(x) = \begin{cases} 0 & \text{if } 0 \leqslant x < 1 \\ 1 & \text{if } x = 1, \end{cases}$$

which is clearly not continuous.

On the other hand, each f_n is continuous, and also bounded since $[0,1]$ is compact, so $f_n([0,1])$ is a compact subset of $\mathbf{R}$, in particular it is bounded. □