Tutorial Week 8

Topics: Pointwise and uniform convergence, approximation, Baire.

8.1. For each $n \in \mathbf{N}$, consider the function $f_n\colon [0,1] \longrightarrow \mathbf{R}$ given by

$$f_n(x) = \frac{x^2}{1+nx}.$$

(a) Prove that f_n is bounded, for all $n \in \mathbf{N}$.

(b) Find the pointwise limit f of the sequence (f_n).

(c) For any $n \in \mathbf{N}$, compute the uniform distance $d_\infty(f_n, f)$.

(d) Does the sequence (f_n) converge uniformly to f?

Solution.

(a) Fix $n \in \mathbf{N}$. If $x \in [0,1]$ then $0 \leqslant x^2 \leqslant 1$ and $1+n \geqslant 1+nx \geqslant 1$, so $1/(1+n) \leqslant 1/(1+nx) \leqslant 1$, so

$$0 \leqslant \frac{x^2}{1+nx} \leqslant 1.$$

Thus f_n is bounded.

(b) For $x = 0$ the sequence $(f_n(x)) = (f_n(0))$ is the constant sequence 0, so $f(0) = 0$.

For $0 < x \leqslant 1$ we have

$$\lim_{n\longrightarrow\infty} \frac{x^2}{1+nx} = x^2 \lim_{n\longrightarrow\infty} \frac{1}{1+nx} = 0,$$

so $f(x) = 0$.

We conclude that the pointwise limit is the constant function $f = 0$ on $[0,1]$.

(c) We have

$$d_\infty(f_n, f) = \sup_{x\in[0,1]} \frac{x^2}{1+nx}.$$

Since f_n is continuous on a compact interval, it attains its extremal values in $[0,1]$; in particular its global maximum is at $x = 0$ or at $x = 1$ or at a stationary point in $(0,1)$.

The derivative is

$$f_n'(x) = \frac{x(2+nx)}{(1+nx)^2},$$

so the stationary points are 0 and $-2/n$, neither of which lies in $(0,1)$. Moreover $f_n(0) = 0$ and $f_n(1) = 1/(1+n)$, so we conclude that

$$d_\infty(f_n, f) = \frac{1}{1+n}.$$

(d) We have $(d_\infty(f_n), f) \longrightarrow 0$ as $n \longrightarrow \infty$, so the convergence is uniform. □

8.2. Let $f_0\colon \mathbf{R} \longrightarrow \mathbf{R}$ be the function defined by

$$f_0(x) = \begin{cases} 1+x & \text{if } -1 \leqslant x \leqslant 0, \\ 1-x & \text{if } 0 < x \leqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

For each positive integer n, define $f_n\colon \mathbf{R} \longrightarrow \mathbf{R}$ by

$$f_n(x) = f_0(x-n).$$

(a) Prove that f_n is bounded, for all $n \in \mathbf{N}$.

(b) Find the pointwise limit f of the sequence (f_n).

(c) For any $n \in \mathbf{N}$, compute the uniform distance $d_\infty(f_n, f)$.

(d) Does the sequence (f_n) converge uniformly to f?

Solution.

(a) It is straightforward to see that $f_n(\mathbf{R}) = [0,1]$ for every natural number n. Thus f_n is bounded.

(b) Fix a real number x and let N be the smallest positive integer such that $x < N$. It follows from the definition of f_n that $f_n(x) = 0$ if $n > N$. Hence $\big(f(x)\big) \longrightarrow 0$ as $n \longrightarrow \infty$ and therefore f is the constant function sending every real number to 0.

(c) We have

$$d_\infty(f_n, f) = \sup_{x \in \mathbf{R}} \big\{ d_{\mathbf{R}}\big(f(x), 0\big) \big\} = d_{\mathbf{R}}\big(f_n(n), 0\big) = 1.$$

(d) Since $d_\infty(f_n, f)$ does not converge to 0, the sequence (f_n) does not converge to f uniformly. □

8.3. Let $X = [0,1] \times [0,1]$ be the unit square with the induced topology from $\mathbf{R}^2$.

Find a subalgebra $\mathcal{A}$ of $C_0(X, \mathbf{R})$ that is dense. (Obviously, try to make $\mathcal{A}$ as small as you can.)

Solution. Let $\mathcal{A} = \mathbf{R}[x, y]$, that is, the algebra of real polynomial functions in two variables x and y (which can be thought of as coordinate projection maps) mapping $X \longrightarrow \mathbf{R}$. By the Stone–Weierstrass theorem (Theorem 2.77), it suffices to prove that $\mathcal{A}$ separates points and is non-vanishing.

Given any two distinct points $(x_1, y_1) \neq (x_2, y_2) \in X$, set $f(x,y) = (x - x_1)^2 + (y - y_1)^2$. Then $f(x_1, y_1) = 0 \neq f(x_2, y_2)$, so $\mathcal{A}$ separates points.

Clearly $f(x, y) = 1$ does not vanish at any point in X, so $\mathcal{A}$ is non-vanishing. □

8.4.

(a) Suppose $f \in C_0([0,1], \mathbf{R})$ has the property that

$$\int_0^1 f(x)\, x^n \, dx = 0 \qquad \text{for all } n = 0, 1, 2, \ldots.$$

Prove that f is the constant function 0 on $[0,1]$.

(b) Give an explicit **discontinuous** function $f \colon [0,1] \longrightarrow \mathbf{R}$ that satisfies the equation in part (a) but is (obviously) not the constant function 0 on $[0,1]$.

Solution.

(a) Let M be an upper bound for $|f|$ on $[0,1]$. If $M = 0$, we are done. So we may assume now that $M > 0$.

Let $\varepsilon > 0$. By the Weierstrass Approximation Theorem there exists $p \in \mathcal{A}$ such that

$$|f(x) - p(x)| < \frac{\varepsilon}{M} \qquad \text{for all } x \in [0,1].$$

Writing $p(x) = \sum_{n=0}^{d} a_n x^n$ with $a_n \in \mathbf{R}$, we have by the linearity of the integral and by the hypothesis in the question:

$$\int_0^1 f(x)p(x)\,dx = \sum_{n=0}^{d} a_n \int_0^1 f(x)\,x^n\,dx = 0.$$

Then

$$\begin{aligned}\left|\int_0^1 f(x)^2\,dx\right| &= \left|\int_0^1 f(x)\big(f(x)-p(x)\big)\,dx\right| \\ &\leqslant \int_0^1 |f(x)||f(x)-p(x)|\,dx \leqslant M\,\frac{\varepsilon}{M} = \varepsilon.\end{aligned}$$

Since this holds for all $\varepsilon > 0$, we conclude that the integral of the non-negative continuous function $f(x)^2$ on $[0,1]$ is zero, hence $f(x)^2$ is the constant function 0 on $[0,1]$, hence so is $f(x)$.

(b) There are many options here, but we can take for instance

$$f(x) = \begin{cases} 0 & \text{if } x \neq \frac{1}{2} \\ 1 & \text{if } x = \frac{1}{2}. \end{cases} \quad \square$$

8.5. Let (X,d) be a nonempty complete metric space. If

$$X = \bigcup_{n\in\mathbf{N}} C_n \quad \text{with each } C_n \text{ a closed subset of X,}$$

then there exists $n \in \mathbf{N}$ such that $C_n^\circ \neq \varnothing$.
[**Hint**: Use the Baire Category Theorem, Theorem 2.79.]

Solution. We proceed by contradiction.

Suppose $C_n^\circ = \varnothing$ for all $n \in \mathbf{N}$. Let $U_n = X \setminus C_n$, then U_n is open and dense in X. By Theorem 2.79,

$$\bigcap_{n\in\mathbf{N}} U_n \quad \text{is dense in } X,$$

in particular it is nonempty. Taking complements, we deduce that $\bigcup_{n\in\mathbf{N}} C_n \neq X$, contradiction. $\square$