Tutorial Week 9

Topics: Norms, inner products

9.1. Let V be an inner product space. Prove that for any $v \in V$ we have

$$\|v\| = \sup_{\|w\|=1} \left|\langle v, w\rangle\right|.$$

Show that the supremum is in fact achieved by a well-chosen w.

Solution. If $v = 0$ then the equality is obvious.

So assume now that $v \neq 0$. By Cauchy–Schwarz we have for all $w \in V$:

$$\left|\langle v, w\rangle\right| \leqslant \|v\| \, \|w\|.$$

Therefore for all $w \in V$ with $\|w\| = 1$ we have

$$\left|\langle v, w\rangle\right| \leqslant \|v\|,$$

so that

$$\sup_{\|w\|=1} \left|\langle v, w\rangle\right| \leqslant \|v\|.$$

To get equality, take $w = \frac{1}{\|v\|} v$ and see that the LHS is indeed $\|v\|$. □

9.2 (Pythagorean theorem)**.** Let $v_1, \dots, v_n$ be pairwise orthogonal vectors in an inner product space V. Prove that

$$\|v_1 + \cdots + v_n\|^2 = \|v_1\|^2 + \cdots + \|v_n\|^2.$$

Solution. It suffices to prove this for two vectors v_1 and v_2 (easy induction follows):

$$\|v_1 + v_2\|^2 = \langle v_1 + v_2, v_1 + v_2\rangle = \langle v_1, v_1\rangle + \langle v_1, v_2\rangle + \langle v_2, v_1\rangle + \langle v_2, v_2\rangle = \|v_1\|^2 + \|v_2\|^2.$$ □

9.3. Let $(V, \|\cdot\|)$ be a normed space.

(a) Fix $u \in V$ and define $T_u \colon V \longrightarrow V$ by $T_u(v) = u + v$ for all $v \in V$. Prove that T_u is a bijective isometry.

(b) Fix $\alpha \in \mathbf{F}^\times$ and define $S_\alpha \colon V \longrightarrow V$ by $S_\alpha(v) = \alpha v$ for all $v \in V$. Prove that S_α is a homeomorphism.

(c) Let $U_1, \dots, U_n$ be nonempty open subsets of V. Let $\alpha_1, \dots, \alpha_n \in \mathbf{F}$ such that at least one $\alpha_i \neq 0$. Let

$$U = \alpha_1 U_1 + \cdots + \alpha_n U_n.$$

Prove that U is a nonempty open subset of V.

Solution.

(a) For all $v_1, v_2 \in V$ we have

$$d(T_u(v_1), T_u(v_2)) = \|T_u(v_1) - T_u(v_2)\| = \|u + v_1 - u - v_2\| = \|v_1 - v_2\| = d(v_1, v_2),$$

So T_u is an isometry. It is also bijective because T_{-u} is its inverse: $T_u \circ T_{-u} = \mathrm{id}_V = T_{-u} \circ T_u$.

(b) We can deduce that S_α is continuous from Proposition 3.3, but we can also prove it directly. Given $\varepsilon > 0$, let $\delta = \varepsilon/|\alpha|$ (this makes sense since $\alpha \in \mathbf{F}^\times$), then for all $v_1, v_2 \in V$ such that $\|v_1 - v_2\| < \delta$ we have

$$\|S_\alpha(v_1) - S_\alpha(v_2)\| = \|\alpha v_1 - \alpha v_2\| = |\alpha|\,\|v_1 - v_2\| < |\alpha|\delta = \varepsilon.$$

So S_α is continuous. It is also bijective because $S_{1/\alpha}$ is its inverse, and $S_{1/\alpha}$ is itself continuous by the above, so S_α is a homeomorphism.

(c) By part (b), if $\alpha_i \neq 0$ and U_i is nonempty open, then $\alpha_i U_i$ is nonempty open. It remains to check that the sum of two nonempty open sets W_1 and W_2 is a nonempty open set:

$$W_1 + W_2 = \{w_1 + w_2 \colon w_1 \in W_1, w_2 \in W_2\} = \bigcup_{w_1 \in W_1} \{w_1 + w_2 \colon w_2 \in W_2\} = \bigcup_{w_1 \in W_1} T_{w_1}(W_2),$$

which is a union of open sets by part (a), hence open. It is clear that $W_1 + W_2$ is nonempty if both W_1 and W_2 are nonempty. $\square$

9.4. Prove that the following norms on $\mathbf{R}^n$ are not defined by inner products:

(a) the ℓ^1-norm defined by

$$\left\|(x_1, \dots, x_n)\right\|_1 = \sum_{i=1}^n |x_i|,$$

(b) the ℓ^∞-norm defined by

$$\left\|(x_1, \dots, x_n)\right\|_\infty = \max\left\{|x_1|, \dots, |x_n|\right\}.$$

Solution. We will verify that neither of the two norms satisfy the Parallelogram Law (Proposition 3.10), so they cannot be defined by inner products. Let $e_1 = (1, 0, \dots, 0)$ and $e_2 = (0, 1, 0, \dots, 0)$.

(a) We have $\|e_1\|_1 = \|e_2\|_1 = 1$ and $\|e_1 + e_2\|_1 = \|e_1 - e_2\|_1 = 2$, so

$$\|e_1 + e_2\|_1^2 + \|e_1 - e_2\|_1^2 = 8 \neq 4 = 2\left(\|e_1\|_1^2 + \|e_2\|_1^2\right).$$

(b) We have $\|e_1\|_\infty = \|e_2\|_\infty = \|e_1 + e_2\|_\infty = \|e_1 - e_2\|_\infty = 1$, so

$$\|e_1 + e_2\|_\infty^2 + \|e_1 - e_2\|_\infty^2 = 2 \neq 4 = 2\left(\|e_1\|_\infty^2 + \|e_2\|_\infty^2\right).$$ $\square$

9.5. Let $\left(V, \|\cdot\|\right)$ be a Banach space. Prove that if a series $\sum_{n=1}^\infty a_n$ in V converges absolutely, then

$$\left\|\sum_{n=1}^\infty a_n\right\| \leqslant \sum_{n=1}^\infty \|a_n\|.$$

(If V is not complete, the result still holds under the extra assumption that the series **converges** in V.)

Solution. If V is Banach, any absolutely convergent series converges, so the left hand side of the desired inequality makes sense.

The claim now follows from the usual triangle inequality. For any $m \in \mathbf{N}$, we have

$$\|a_1 + \cdots + a_m\| \leqslant \|a_1\| + \cdots + \|a_m\|.$$

Taking limits as $m \longrightarrow \infty$ we get

$$\left\|\sum_{n=1}^\infty a_n\right\| = \left\|\lim_{m\longrightarrow\infty} \sum_{n=1}^m a_n\right\| = \lim_{m\longrightarrow\infty} \left\|\sum_{n=1}^m a_n\right\| \leqslant \lim_{m\longrightarrow\infty} \sum_{n=1}^m \|a_n\| = \sum_{n=1}^\infty \|a_n\|.$$ $\square$

9.6. Prove that every finite-dimensional normed vector space V is separable.

More precisely, fix a basis $\{v_1, \ldots, v_n\}$ of V and let

$$\mathbf{K} = \begin{cases} \mathbf{Q} & \text{if } \mathbf{F} = \mathbf{R} \\ \mathbf{Q}[i] & \text{if } \mathbf{F} = \mathbf{C} \end{cases} \qquad \text{and } D = \mathrm{Span}_{\mathbf{K}}\{v_1, \ldots, v_n\}.$$

Prove that D is a dense countable subset of V.

Solution. We start by noting that $\mathbf{K}$ is dense in $\mathbf{F}$. We already know this for $\mathbf{Q}$ in $\mathbf{R}$. For $\mathbf{Q}[i]$ in $\mathbf{C} = \mathbf{R}[i]$: given $z = x + iy$ and $\varepsilon > 0$, let $p, q \in \mathbf{Q}$ be such that $|p - x| < \varepsilon/2$ and $|q - y| < \varepsilon/2$. Then

$$|(p + iq) - z| = |(p - x) + i(q - y)| \leqslant |p - x| + |q - y| < \varepsilon.$$

By Theorem 3.7, the norm on V is equivalent to the norm $\|\cdot\|_1$ defined by

$$\|\alpha_1 v_1 + \cdots + \alpha_n v_n\|_1 = |\alpha_1| + \cdots + |\alpha_n|.$$

Since separability is a property of topological spaces and equivalent norms give rise to the same topology (see Exercise 2.2), we can assume without loss of generality that V is equipped with the norm $\|\cdot\|_1$.

Let $v \in V$ and $\varepsilon > 0$. We have

$$v = \alpha_1 v_1 + \cdots + \alpha_n v_n, \qquad \alpha_i \in \mathbf{F}.$$

For each i, there exists $\beta_i \in \mathbf{K}$ such that $|\alpha_i - \beta_i| < \varepsilon/n$. Define

$$v' = \beta_1 v_1 + \cdots + \beta_n v_n \in D.$$

We have

$$\|v - v'\| = \|(\alpha_1 - \beta_1)v_1 + \cdots + (\alpha_n - \beta_n)v_n\|_1 = |\alpha_1 - \beta_1| + \cdots + |\alpha_n - \beta_n| < \varepsilon.$$

The set D is countable because it is the Cartesian product of n copies of the countable field $\mathbf{K}$. $\square$