

Tutorial Week 10

Topics: bounded linear transformations, convexity, sequence spaces.

10.1. Let $X = C_0([0, 1], \mathbf{R})$ be the Banach space of continuous functions

$$f: [0, 1] \longrightarrow \mathbf{R}$$

with the supremum norm, as described in [Proposition 3.13](#).

Define $\phi: X \longrightarrow \mathbf{R}$ by $\phi(f) = f(0)$ for all $f \in X$.

Prove that ϕ is a bounded linear map.

Solution. It is clear that ϕ is linear:

$$\phi(f + g) = (f + g)(0) = f(0) + g(0) = \phi(f) + \phi(g)$$

and

$$\phi(\alpha f) = (\alpha f)(0) = \alpha f(0) = \alpha \phi(f).$$

It is also clearly bounded:

$$|\phi(f)| = |f(0)| \leq \|f\|,$$

as $\|f\|$ is the supremum of $|f(x)|$ for $x \in [0, 1]$. □

10.2. Let V, W be normed spaces over $\mathbf{F}$. Let C be a convex subset of V and D a convex subset of W . Prove that $C \times D$ is a convex subset of $V \times W$.

Solution. This is straightforward: let $(c_1, d_1), (c_2, d_2) \in C \times D$ and let $t \in [0, 1]$. Then

$$t(c_1, d_1) + (1 - t)(c_2, d_2) = (tc_1 + (1 - t)c_2, td_1 + (1 - t)d_2) \in C \times D,$$

as $tc_1 + (1 - t)c_2 \in C$ (since C is convex), and similarly $td_1 + (1 - t)d_2 \in D$. □

10.3 (Minkowski's Inequality for ℓ^∞). Prove that if $u, v \in \ell^\infty$, then

$$\|u + v\|_{\ell^\infty} \leq \|u\|_{\ell^\infty} + \|v\|_{\ell^\infty}.$$

In particular, $u + v \in \ell^\infty$.

Solution. We have

$$\|u + v\|_{\ell^\infty} = \sup_{n \in \mathbf{N}} \{|u_n + v_n|\} \leq \sup_{n \in \mathbf{N}} \{|u_n| + |v_n|\} \leq \sup_{n \in \mathbf{N}} |u_n| + \sup_{n \in \mathbf{N}} |v_n| = \|u\|_{\ell^\infty} + \|v\|_{\ell^\infty}. \quad \square$$

10.4 (Hölder's Inequality for ℓ^1 and ℓ^∞). Prove that if $u = (u_n) \in \ell^\infty$ and $v = (v_n) \in \ell^1$, then

$$\sum_{n=1}^{\infty} |u_n v_n| \leq \|u\|_{\ell^\infty} \|v\|_{\ell^1}.$$

Solution. By the definition of ℓ^∞ and the ℓ^∞ -norm, we have $|u_n| \leq \|u\|_{\ell^\infty}$ for all $n \in \mathbf{N}$. Therefore for any $m \in \mathbf{N}$ we have

$$\sum_{n=1}^m |u_n v_n| \leq \|u\|_{\ell^\infty} \sum_{n=1}^m |v_n|,$$

but the latter series converges because $v \in \ell^1$, to $\|v\|_{\ell^1}$ and we get

$$\sum_{n=1}^{\infty} |u_n v_n| \leq \|u\|_{\ell^\infty} \|v\|_{\ell^1}. \quad \square$$

10.5. Suppose $1 \leq p \leq q$. Prove that

$$\ell^p \subseteq \ell^q.$$

Show that if $p < q$ then the inclusion is strict: $\ell^p \subsetneq \ell^q$.

Solution. We prove that

$$\|x\|_{\ell^q} \leq \|x\|_{\ell^p} \quad \text{for all } x \in \ell^p.$$

If $\|x\|_{\ell^p} = 0$ then $x = 0$ so $\|x\|_{\ell^q} = 0$ and the inequality obviously holds. So suppose $x \neq 0$, then by dividing through by $\|x\|_{\ell^p}$ we can reduce to proving that

$$\|x\|_{\ell^q} \leq 1 \quad \text{for all } x \text{ such that } \|x\|_{\ell^p} = 1.$$

But if $\|x\|_{\ell^p} = 1$ then

$$\sum_{n=1}^{\infty} |x_n|^p = 1,$$

which means that for all $n \in \mathbf{N}$ we have $|x_n|^p \leq 1$, so $|x_n| \leq 1$. However, $p \leq q$ and $|x_n| \leq 1$ implies that $|x_n|^q \leq |x_n|^p$ for all $n \in \mathbf{N}$, so that

$$\|x\|_{\ell^q}^q = \sum_{n=1}^{\infty} |x_n|^q \leq \sum_{n=1}^{\infty} |x_n|^p = 1.$$

If $p < q$ then $\alpha := q/p > 1$. For each $n \in \mathbf{N}$, let

$$x_n = \frac{1}{n^{1/p}},$$

so that

$$|x_n|^p = \frac{1}{n}, \quad |x_n|^q = \frac{1}{n^\alpha}.$$

We have

$$\|(x_n)\|_{\ell^p} = \sum_{n=1}^{\infty} \frac{1}{n} = \infty, \quad \|(x_n)\|_{\ell^q} = \sum_{n=1}^{\infty} \frac{1}{n^\alpha} < \infty,$$

so $(x_n) \in \ell^q \setminus \ell^p$. □

10.6.

(a) Prove that the function $f: \ell^1 \rightarrow \mathbf{F}$ defined by

$$f((a_n)) = \sum_{n=1}^{\infty} a_n.$$

is continuous.

(b) Prove that the following subset is a closed subspace of ℓ^1 :

$$S = \left\{ (a_n) \in \ell^1 : \sum_{n=1}^{\infty} a_n = 0 \right\}.$$

Solution. (a) First note that this is a reasonable definition, because the infinite series on the right hand side converges in $\mathbf{F}$:

$$\left| \sum_{n=1}^N a_n \right| \leq \sum_{n=1}^N |a_n|,$$

and the latter converges as $N \rightarrow \infty$ since $(a_n) \in \ell^1$.

The function f is linear. It is also bounded, because as we have just seen:

$$|f((a_n))| = \left| \sum_{n=1}^{\infty} a_n \right| \leq \sum_{n=1}^{\infty} |a_n| = \|(a_n)\|_{\ell^1}.$$

Hence f is continuous by [Proposition 3.16](#).

- (b) As the kernel of the continuous linear functional f , the subset S of ℓ^1 is a closed subspace. $\square$

10.7. Consider the left shift map $L: \mathbf{F}^{\mathbf{N}} \longrightarrow \mathbf{F}^{\mathbf{N}}$ given by $L((a_n)) = (a_{n+1})$, that is

$$L(a_1, a_2, a_3, \dots) = (a_2, a_3, \dots).$$

- (a) Prove that L is a surjective linear map. What is the kernel of L ?
- (b) Prove that for all $1 \leq p \leq \infty$, the restriction of L to ℓ^p is a surjective continuous map onto ℓ^p .
- (c) Define the right shift map $R: \mathbf{F}^{\mathbf{N}} \longrightarrow \mathbf{F}^{\mathbf{N}}$ and prove that it is an injective linear map, the restriction of which is an isometry for any ℓ^p with $1 \leq p \leq \infty$.
- (d) Check that $L \circ R = \text{id}_{\mathbf{F}^{\mathbf{N}}} \neq R \circ L$.

Solution.

- (a) It is clear that L is surjective. Linearity is pretty straightforward, and it's also clear that $\ker(L) = \text{Span}\{e_1\}$.
- (b) We have

$$\|L(a_1, a_2, a_3, \dots)\|_{\ell^p} = \left(\sum_{n=2}^{\infty} |a_n|^p \right)^{1/p} \leq \left(\sum_{n=1}^{\infty} |a_n|^p \right)^{1/p} = \|(a_1, a_2, \dots)\|_{\ell^p},$$

so L is bounded, and $L((a_n)) \in \ell^p$ if $(a_n) \in \ell^p$.

For the surjectivity note that if $b = (b_1, b_2, \dots) \in \ell^p$, then

$$b = L(a) \quad \text{for } a = (0, b_1, b_2, \dots)$$

and $\|a\|_{\ell^p} = \|b\|_{\ell^p}$, so $a \in \ell^p$.

The case of ℓ^∞ is done in a similar way.

- (c) To get a linear map we need to set

$$R(a_1, a_2, a_3, \dots) = (0, a_1, a_2, a_3, \dots).$$

Both injectivity and linearity are straightforward.

We have, for $p \geq 1$ or $p = \infty$:

$$\|R(a_1, a_2, \dots)\|_{\ell^p} = \|(0, a_1, a_2, \dots)\|_{\ell^p} = \|(a_1, a_2, \dots)\|_{\ell^p},$$

so R is an isometry and $R(a) \in \ell^p$ if $a \in \ell^p$.

- (d) Clear. For any $a = (a_n) \in \mathbf{F}^{\mathbf{N}}$ we have

$$L(R(a)) = L(R(a_1, a_2, \dots)) = L(0, a_1, a_2, \dots) = (a_1, a_2, \dots) = a,$$

$$R(L(a)) = R(L(a_1, a_2, \dots)) = R(a_2, a_3, \dots) = (0, a_2, a_3, \dots) \neq a \text{ unless } a_1 = 0. \quad \square$$