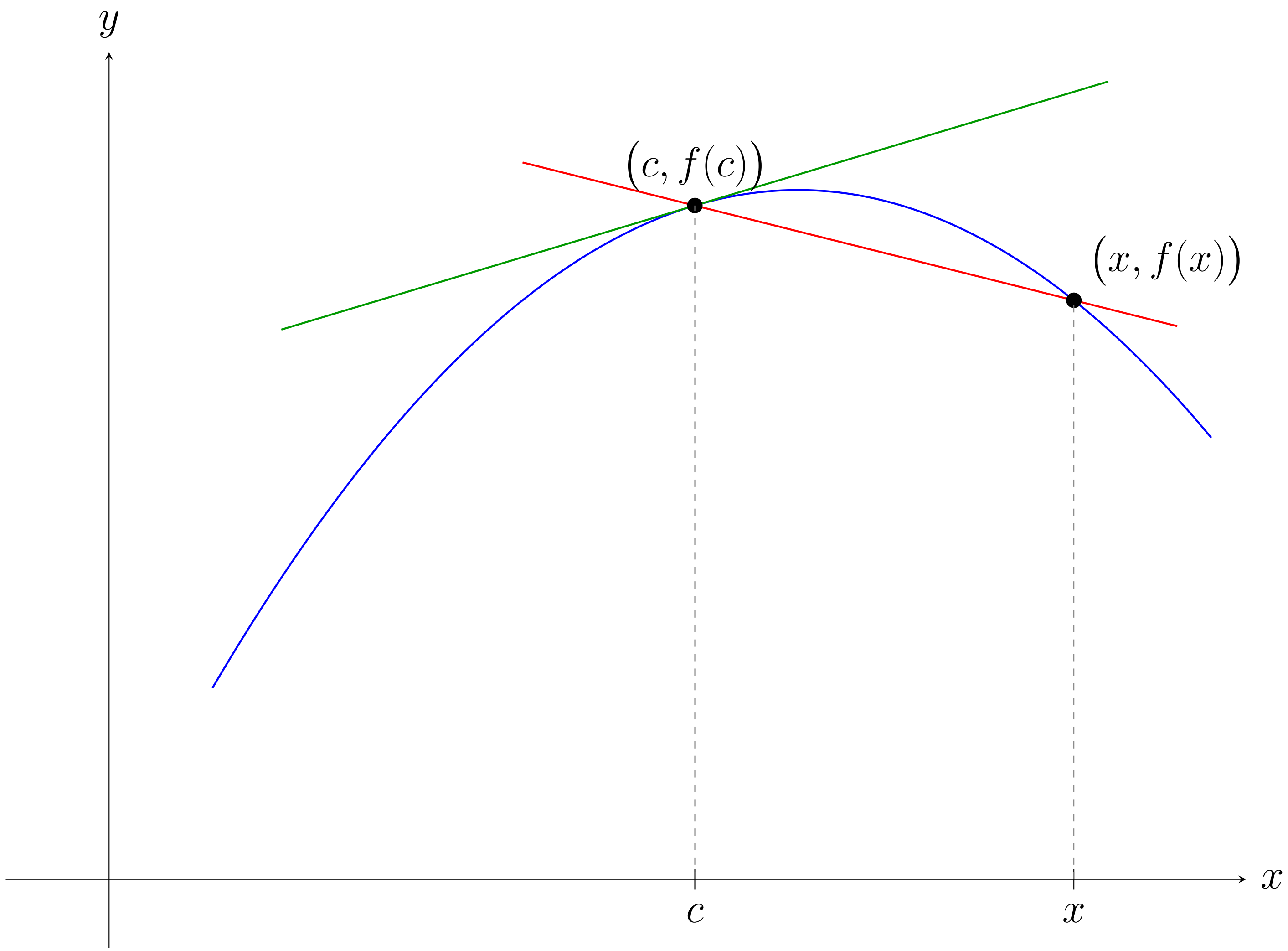
# 5 Differentiation and integration

## Chapter contents

# 5.1 Derivatives



We can easily find the slope of any secant line:

$$s_c(x) = \frac{f(x) - f(c)}{x - c}.$$

The idea is then to take the limit of this function $s_c(x)$ as $x \longrightarrow c$.

The geometric definition of the derivative of $f$ at $c$ is: the slope of the tangent line to the curve $y = f(x)$ at the point $\big(c, f(c)\big)$.

**Definition 5.1.** Let $I \subseteq \mathbf{R}$ be an interval, $f: I \longrightarrow \mathbf{R}$ a function, $c \in I$.

We say that $f$ is *differentiable at $c$* if there exists $L_c \in \mathbf{R}$ such that

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = L_c.$$

The number $L_c$ is called the *derivative of $f$ at $c$*.

If $f$ is differentiable at $c$ for all $c \in I$, we say that $f$ is *differentiable on $I$*; the *derivative of $f$* is the function $f': I \longrightarrow \mathbf{R}$ defined by $f'(c) = L_c$.

By letting $h = x - c$, we can the derivative of $f$ at $c$ as

$$\lim_{h \to 0} \frac{f(c+h) - f(c)}{h} = L_c.$$

**Example 5.2.** The function $f: \mathbf{R} \longrightarrow \mathbf{R}$ given by $f(x) = x^2$ is differentiable on $\mathbf{R}$.

Let $c \in \mathbf{R}$, then

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} \frac{x^2 - c^2}{x - c} = \lim_{x \to c} (x + c) = 2c.$$

Therefore $f$ is differentiable at every $c \in \mathbf{R}$, with derivative $f'(c) = 2c$.

**Example 5.3.** The function $g:\ \mathbf{R} \longrightarrow \mathbf{R}$ given by $g(x) = |x|$ is not differentiable at $0$.

Sketch graph of absolute value function.

We have

$$\begin{aligned}
\lim_{x\to 0^-} \frac{g(x)-g(0)}{x} &= \lim_{x\to 0^-} \frac{|x|}{x} = \lim_{x\to 0^-} \frac{-x}{x} = \lim_{x\to 0^-} (-1) = -1,\\
\lim_{x\to 0^+} \frac{g(x)-g(0)}{x} &= \lim_{x\to 0^+} \frac{|x|}{x} = \lim_{x\to 0^+} \frac{x}{x} = \lim_{x\to 0^+} (1) = 1.
\end{aligned}$$

Since the two one-sided limits do not agree, the limit as $x \longrightarrow 0$ does not exist, so $g$ is not differentiable at $0$.

**Theorem 5.4.** *Let $I \subseteq \mathbf{R}$ be an interval, $f : I \longrightarrow \mathbf{R}$ a function, $c \in I$. If $f$ is differentiable at $c$, then $f$ is continuous at $c$.*

*Proof.* We want to show that

$$\lim_{x\to c} f(x) = f(c), \quad \text{that is } \lim_{x\to c} \big(f(x) - f(c)\big) = 0.$$

Since $f$ is differentiable at $c$, there exists $L \in \mathbf{R}$ such that

$$\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = L.$$

Therefore

$$\lim_{x\to c} \big(f(x)-f(c)\big) = \lim_{x\to c} \left(\frac{f(x)-f(c)}{x-c}(x-c)\right) = \left(\lim_{x\to c} \frac{f(x)-f(c)}{x-c}\right)\left(\lim_{x\to c}(x-c)\right) = L \cdot 0 = 0.$$

We conclude that $f$ is continuous at $c$. □

**Theorem 5.5** (Linearity)**.** *Let $I \subseteq \mathbf{R}$ be an interval, $c \in I$, $f, g : I \longrightarrow \mathbf{R}$ two functions that are differentiable at $c$. Then*

- *$f + g : I \longrightarrow \mathbf{R}$ is differentiable at $c$ and $(f+g)'(c) = f'(c) + g'(c)$;*
- *for any $\lambda \in \mathbf{R}$, $\lambda f : I \longrightarrow \mathbf{R}$ is differentiable at $c$ and $(\lambda f)'(c) = \lambda f'(c)$.*

Let $\mathsf{Fun}$ be the real vector space of all functions $\mathbf{R} \longrightarrow \mathbf{R}$; let $\mathsf{Diff}_I$ be the subset of functions that are differentiable on $I$.

The Theorem says that $\mathsf{Diff}_I$ is a subspace of $\mathsf{Fun}$, and that the map $T : \mathsf{Diff}_I \longrightarrow \mathsf{Fun}$ given by $T(f) = f'$ is a linear transformation.

You get to prove this in Exercise 5.1.

**Theorem 5.6** (Product Rule)**.** *Let $I \subseteq \mathbf{R}$ be an interval, $c \in I$, $f, g\colon\ I \longrightarrow \mathbf{R}$ two functions that are differentiable at $c$. Then the product $fg\colon\ I \longrightarrow \mathbf{R}$ is differentiable at $c$ and*

$$(fg)'(c) = f'(c)g(c) + f(c)g'(c).$$

*Proof.* We have

$$\begin{aligned}(fg)'(c) &= \lim_{x\to c} \frac{(f(x)g(x) - f(c)g(c)}{x-c} = \lim_{x\to c} \frac{(f(x)-f(c))g(x) + f(c)(g(x)-g(c))}{x-c} \\ &= \left(\lim_{x\to c} \frac{f(x)-f(c)}{x-c}\right) \lim_{x\to c} g(x) + f(c)\left(\lim_{x\to c} \frac{g(x)-g(c)}{x-c}\right) = f'(c)g(c) + f(c)g'(c).\end{aligned}$$

Here we used the Algebra of Limits throughout; we also used Theorem 5.4 to see that $g$ is continuous at $c$. ☐

**Example 5.7.** For each $n \in \mathbf{Z}_{\geqslant 1}$ the function $f : \mathbf{R} \longrightarrow \mathbf{R}$ given by $f(x) = x^n$ is differentiable on $\mathbf{R}$ with derivative $nx^{n-1}$.

We can prove this by induction.

For the base case $n = 1$, the function is $f(x) = x$: let $c \in \mathbf{R}$, then

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c} \frac{x - c}{x - c} = \lim_{x \to c}(1) = 1,$$

so the function $f(x) = x$ is differentiable at $c$ and its derivative at $c$ is $f'(c) = 1 = 1 \cdot x^{1-1}$.

For the induction step: fix $k \geqslant 1$ and assume the statement holds for the function $g(x) = x^k$. Consider the function $h(x) = x^{k+1} = f(x)g(x)$. By Theorem 5.6, $h$ is differentiable at $c$ and its derivative is

$$h'(c) = f'(c)g(c) + f(c)g'(c) = c^k + ckc^{k-1} = (k+1)c^k.$$

One of the big wins of derivatives is that they can be used to optimise functions.

**Definition 5.8.** Let $f : [a,b] \longrightarrow \mathbf{R}$ be a function, $c \in (a,b)$.

- We say that $f$ has a *local maximum* at $c$ if there exists $\delta > 0$ such that

$$\text{for all } x \in [a,b], \text{ if } |x-c| < \delta \text{ then } f(x) \leqslant f(c).$$

- We say that $f$ has a *local minimum* at $c$ if there exists $\delta > 0$ such that

$$\text{for all } x \in [a,b], \text{ if } |x-c| < \delta \text{ then } f(x) \geqslant f(c).$$

Recall that if $f$ is continuous on $[a, b]$, it attains its extremal values on $[a, b]$ (by the Extremal Value Theorem).

**Theorem 5.9.** *Let $f : [a, b] \longrightarrow \mathbf{R}$ be continuous, and suppose that $f$ is differentiable at some $c \in (a, b)$. If $f$ has a local maximum or minimum at $c$, then $f'(c) = 0$.*

*Proof.* We prove the statement for a local maximum (the other case is very similar).

Suppose $f$ has a local maximum at $c \in (a, b)$, then there exists $\delta > 0$ such that for all $x \in [a, b]$, if $|x - c| < \delta$ then $f(x) \leqslant f(c)$.

Since $f$ is differentiable at $c$ we have

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c}.$$

If $c - \delta < x < c$ then $f(x) \leqslant f(c)$, so $f(x) - f(c) \leqslant 0$ and $x - c < 0$, so

$$f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \geqslant 0.$$

If $c < x < c + \delta$ then we still have $f(x) - f(c) \leqslant 0$, but now $x - c > 0$, so

$$f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \leqslant 0.$$

Since $f'(c) \geqslant 0$ and $f'(c) \leqslant 0$, we conclude that $f'(c) = 0$. □

Recall that the Intermediate Value Theorem says that if $f : [a, b] \longrightarrow \mathbf{R}$ is continuous on $[a, b]$, then $f$ takes all values between $f(a)$ and $f(b)$.

If we know that $f$ is differentiable on $[a, b]$, we can say something about the values of the derivative $f'$ on the interval $[a, b]$:

**Theorem 5.10** (Mean Value Theorem)**.** *Let* $a < b$*,* $f : [a, b] \longrightarrow \mathbf{R}$ *be continuous on* $[a, b]$ *and differentiable on* $(a, b)$*. There exists* $c \in (a, b)$ *such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

**Example 5.11.** If you drive 6 km in 5 minutes on a road with a speed limit of 60 km/h, should you get a speeding ticket?

Yes. Your average speed is 72 km/h, so by the Mean Value Theorem there is at least one point in your journey where your speed is 72 km/h. This is a fine of $407, and 3 demerit points on your license.

We start by proving a special case of the Mean Value Theorem:

**Theorem 5.12** (Rolle)**.** *Let* $a < b$, $f : [a, b] \longrightarrow \mathbf{R}$ *be continuous on* $[a, b]$ *and differentiable on* $(a, b)$*. If* $f(a) = f(b) = 0$*, then there exists* $c \in (a, b)$ *such that* $f'(c) = 0$*.*

*Proof.* By the Extreme Value Theorem, there exist $k, \ell \in [a, b]$ such that

$$f(k) \leqslant f(x) \leqslant f(\ell) \qquad \text{for all } x \in [a, b].$$

There are three possibilities:

- $f(k) = f(\ell) = 0$.
  Then $f(x) = 0$ for all $x \in [a, b]$, hence $f'(x) = 0$ for all $x \in (a, b)$.
- $f(k) \neq 0$.
  In particular, $k \neq a, b$, so $k \in (a, b)$ and $f$ has a local minimum at $k$. Hence by Theorem 5.9 we get that $f'(k) = 0$.
- $f(\ell) \neq 0$.
  Similar to the above, we get that $f'(\ell) = 0$. ☐

Now we can prove the Mean Value Theorem:

*Proof of Theorem 5.10.* Let $s : [a, b] \longrightarrow \mathbf{R}$ be given by

$$s(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a).$$

Since $s$ is a polynomial function (of degree one), it is continuous on $[a, b]$ and differentiable on $(a, b)$.

Let $h : [a, b] \longrightarrow \mathbf{R}$ be given by $h(x) = f(x) - s(x)$. Then $h$ is continuous on $[a, b]$ and differentiable on $(a, b)$.

We have

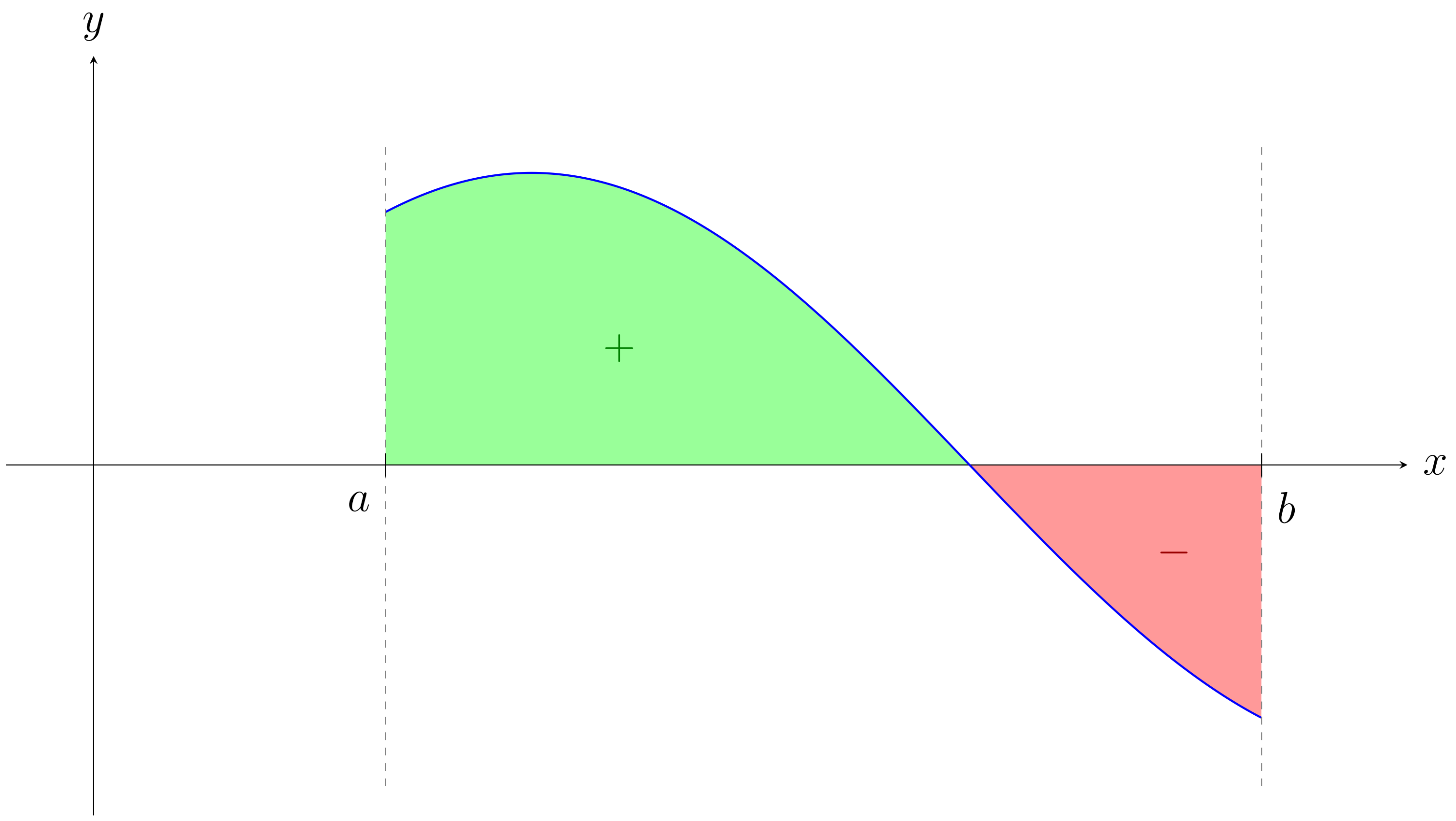
$$h(a) = f(a) - s(a) = f(a) - f(a) = 0, \qquad h(b) = f(b) - s(b) = f(b) - f(b) = 0,$$

so we can apply Rolle's Theorem (Theorem 5.12) to $h$: there exists $c \in (a, b)$ such that $h'(c) = 0$. But $h'(c) = f'(c) - s'(c)$, so

$$f'(c) = s'(c) = \frac{f(b) - f(a)}{b - a}.$$

□

# 5.2 Integrals



The geometric definition of the definite integral of a function $f$ on a closed interval $[a, b]$ of finite length is the area of the region between the graph of $f$, the $x$-axis, and the vertical lines $x = a$ and $x = b$, where areas above the $x$-axis are taken with a positive sign, and areas below the $x$-axis with a negative sign.

**Definition 5.13.** A *partition* of the interval $[a, b]$ is a set

$$P = \{x_0, x_1, \ldots, x_{n-1}, x_n\},$$

where $n \in \mathbf{Z}_{\geqslant 1}$ and

$$a = x_0 \leqslant x_1 \leqslant x_2 \leqslant \ldots \leqslant x_{n-1} \leqslant x_n = b.$$

The intervals $[x_{i-1}, x_i]$ are called *subintervals* of $[a, b]$.

**Example 5.14.** $P = \{3, 4, 5, 6\}$ is a partition of $[3, 6]$ into subintervals

$$[3, 4], \qquad [4, 5], \qquad [5, 6].$$

$Q = \{3, \pi, \pi + 1, 4.5, 6\}$ is a partition of $[3, 6]$ into subintervals

$$[3, \pi], \qquad [\pi, \pi + 1], \qquad [\pi + 1, 4.5], \qquad [4.5, 6].$$

**Definition 5.15.** Let $f : [a, b] \longrightarrow \mathbf{R}$ be bounded on $[a, b]$ and let $P = \{x_0, \ldots, x_n\}$ be a partition of $[a, b]$.

The *lower Riemann sum* of $f$ with respect to $P$ is

$$L(f, P) = \sum_{k=1}^{n} m_k(x_k - x_{k-1}) = m_1(x_1 - x_0) + m_2(x_2 - x_1) + \cdots + m_n(x_n - x_{n-1}),$$

where for $k = 1, \ldots, n$ we have

$$m_k = \inf\{f(x) : \ x \in [x_{k-1}, x_k]\}.$$

The *upper Riemann sum* of $f$ with respect to $P$ is

$$U(f, P) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}) = M_1(x_1 - x_0) + M_2(x_2 - x_1) + \cdots + M_n(x_n - x_{n-1}),$$

where for $k = 1, \ldots, n$ we have

$$M_k = \sup\{f(x) : \ x \in [x_{k-1}, x_k]\}.$$

**Example 5.16.** Find lower and upper bounds for the area under $f: [1,4] \longrightarrow \mathbf{R}$, $f(x) = 25 - x^2$, for $1 \leqslant x \leqslant 4$.

Consider the partition $P = \{1,2,3,4\}$ of $[1,4]$. We have

$$\begin{aligned} m_1 &= \inf\{f(x) : x \in [1,2]\} = f(2) = 21 & \qquad M_1 &= \sup\{f(x) : x \in [1,2]\} = f(1) = 24 \\ m_2 &= \inf\{f(x) : x \in [2,3]\} = f(3) = 16 & \qquad M_2 &= \sup\{f(x) : x \in [2,3]\} = f(2) = 21 \\ m_3 &= \inf\{f(x) : x \in [3,4]\} = f(4) = 9 & \qquad M_3 &= \sup\{f(x) : x \in [3,4]\} = f(3) = 16. \end{aligned}$$

Therefore

$$\begin{aligned} L(f,P) &= m_1(2-1) + m_2(3-2) + m_3(4-3) = 21 + 16 + 9 = 46, \\ U(f,P) &= M_1(2-1) + M_2(3-2) + M_3(4-3) = 24 + 21 + 16 = 61, \end{aligned}$$

so that the area is between 46 and 61.

**Lemma 5.17.** *Let $f : [a,b] \longrightarrow \mathbf{R}$ be bounded on $[a,b]$. For every partition $P$ of $[a,b]$ we have*

$$L(f,P) \leqslant U(f,P).$$

*Proof.* Write $P = \{x_0, x_1, \ldots, x_{n-1}, x_n\}$. For each $1 \leqslant k \leqslant n$ we have $m_k \leqslant M_k$, therefore

$$L(f,P) = \sum_{k=1}^{n} m_k(x_k - x_{k-1}) \leqslant \sum_{k=1}^{n} M_k(x_k - x_{k-1}) = U(f,P).$$ ☐

It feels like more should be true though: it should be the case that any lower sum (with respect to any partition) should be less than any upper sum (with respect to any other partition). This takes a bit more work.

**Definition 5.18.** If $P$ and $Q$ are partitions of $[a, b]$, we say that $Q$ is a *refinement* of $P$ if $P \subseteq Q$.

**Example 5.19.** For the interval $[1, 5]$, $Q = \{1, 2.5, 3, \pi, 4, 4.1, 5\}$ is a refinement of the partition $P = \{1, 3, 4, 5\}$.

**Lemma 5.20.** *Let* $f : [a,b] \longrightarrow \mathbf{R}$ *be bounded on* $[a,b]$*,* $R$ *a partition of* $[a,b]$*, and* $Q$ *a refinement of* $R$ *such that* $Q \setminus R$ *has a single element. Then*

$$L(f,R) \leqslant L(f,Q) \qquad \text{and} \qquad U(f,Q) \leqslant U(f,R).$$

*Proof.* Let $t \in \mathbf{R}$ be such that $Q \setminus R = \{t\}$. Fix $j$ such that $t \in (x_{j-1}, x_j)$. Let

$$m_t^- = \inf\{f(x) : \; x \in [x_{j-1}, t]\}, \qquad m_t^+ = \inf\{f(x) : \; x \in [t, x_j]\}.$$

Then we have $m_j \leqslant m_t^-$ and $m_j \leqslant m_t^+$ and

$$m_j(x_j - x_{j-1}) = m_j(t - x_{j-1}) + m_j(x_j - t) \leqslant m_t^-(t - x_{j-1}) + m_t^+(x_j - t).$$

Therefore

$$\begin{aligned} L(f,R) &= \sum_{k=1}^{j-1} m_k(x_k - x_{k-1}) + m_j(x_j - x_{j-1}) + \sum_{k=j+1}^{n} m_k(x_k - x_{k-1}) \\ &\leqslant \sum_{k=1}^{j-1} m_k(x_k - x_{k-1}) + m_t(t - x_{j-1}) + m_t'(x_j - t) + \sum_{k=j+1}^{n} m_k(x_k - x_{k-1}) \\ &= L(f,Q). \end{aligned}$$

□

**Lemma 5.21.** *Let $f : [a, b] \longrightarrow \mathbf{R}$ be bounded on $[a, b]$, $P$ a partition of $[a, b]$, and $Q$ a refinement of $P$. Then*

$$L(f, P) \leqslant L(f, Q) \qquad \text{and} \qquad U(f, Q) \leqslant U(f, P).$$

*Proof.* We prove the statement for the lower Riemann sums $L$; the upper sums $U$ are very similar.

We proceed by induction on $m$, the cardinality of the set $Q \setminus P$.

The base case is $m = 0$: here $Q = P$ so trivially $L(f, P) \leqslant L(f, Q)$.

The induction step: fix $m \geqslant 0$ and suppose the statement is true for this value of $m$. In other words, we suppose that if $R$ is any refinement of $P$ such that $R \setminus P$ has $m$ elements, then $L(f, P) \leqslant L(f, R)$.

Let $Q$ be a refinement of $P$ such that $Q \setminus P$ has $m + 1$ elements. Fix $t \in Q \setminus P$ and let $R = Q \setminus \{t\}$. By Lemma 5.20, we have $L(f, R) \leqslant L(f, Q)$. Using the induction hypothesis, we conclude that $L(f, P) \leqslant L(f, Q)$. ☐

**Theorem 5.22.** *Let* $f : [a,b] \longrightarrow \mathbf{R}$ *be bounded on* $[a,b]$ *and* $P, P'$ *partitions of* $[a,b]$*. Then*

$$L(f,P) \leqslant U(f,P').$$

*Proof.* Let $Q = P \cup P'$, then $Q$ is a refinement of both $P$ and $P'$. By Lemmas 5.17 and 5.21 we have

$$L(f,P) \leqslant L(f,Q) \leqslant U(f,Q) \leqslant U(f,P'). \qquad \square$$

**Definition 5.23.** Let $f: [a,b] \longrightarrow \mathbf{R}$ be bounded on $[a,b]$.

The *lower Riemann integral* of $f$ on $[a,b]$ is

$$L(f) = \sup\{L(f,P) : \ P \text{ is a partition of } [a,b]\}.$$

The *upper Riemann integral* of $f$ on $[a,b]$ is

$$U(f) = \inf\{U(f,P) : \ P \text{ is a partition of } [a,b]\}.$$

**Theorem 5.24.** *Let $f : [a,b] \longrightarrow \mathbf{R}$ be bounded on $[a,b]$. Then*

$$L(f) \leqslant U(f).$$

*Proof.* Let $P$ and $P'$ be arbitrary partitions of $[a,b]$. By Theorem 5.22 we have

$$U(f,P') \geqslant L(f,P).$$

Since this holds for all partitions $P'$ of $[a,b]$, we get that $L(f,P)$ is a lower bound for the set $\{U(f,P') : \ P' \text{ partition of } [a,b]\}$. Since $U(f)$ is the **greatest** lower bound of this set, we get

$$U(f) \geqslant L(f,P).$$

Since this holds for all partitions $P$ of $[a,b]$, we get that $U(f)$ is an upper bound for the set $\{L(f,P) : \ P \text{ partition of } [a,b]\}$. Since $L(f)$ is the **least** upper bound of this set, we get

$$U(f) \geqslant L(f).$$

□

But surely we actually have equality $L(f) = U(f)$?

**Example 5.25.** Consider the function $f : [1, 2] \longrightarrow \mathbf{R}$ given by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbf{Q}, \\ 0 & \text{if } x \notin \mathbf{Q}. \end{cases}$$

Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[1, 2]$. Since $\mathbf{Q}$ is dense in $\mathbf{R}$ (Corollary 2.61), each interval contains a rational number, where $f$ takes value 1. Since $\mathbf{R} \setminus \mathbf{Q}$ is dense in $\mathbf{R}$ (we haven't proved this yet), each interval contains an irrational number, where $f$ takes value 0. Hence $m_i = 0$ and $M_i = 1$ for all $i = 1, \ldots, n$. Therefore $L(f, P) = 0$ and $U(f, P) = 1$.

As we range over all partitions $P$ of $[1, 2]$ we get $L(f) = 0 < 1 = U(f)$.

**Definition 5.26.** Let $f: [a,b] \longrightarrow \mathbf{R}$ be bounded on $[a,b]$.

We say that $f$ is *Riemann integrable* on $[a,b]$ if $L(f) = U(f)$, in which case we define the *Riemann integral* of $f$ on $[a,b]$ to be

$$\int_a^b f(x)\,dx = L(f) = U(f).$$

Example 5.25 exhibited a non-integrable function.

**Theorem 5.27.** *Let $f: [a,b] \longrightarrow \mathbf{R}$ be bounded on $[a,b]$. The function $f$ is Riemann integrable on $[a,b]$ if and only if for every $\varepsilon > 0$ there exists a partition $P_\varepsilon$ of $[a,b]$ such that*

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon.$$

*Proof.* Suppose $f$ is Riemann integrable on $[a,b]$ and let $\varepsilon > 0$.

We have $L(f) + \varepsilon > L(f) = U(f)$, so $L(f) + \varepsilon$ is not a lower bound for $\{U(f,P): \ P\}$. Therefore there exists a partition $P_\varepsilon^{'}$ such that $U(f, P_\varepsilon^{'}) < L(f) + \varepsilon$.

This means that $U(f, P_\varepsilon^{'}) - \varepsilon < L(f)$, so $U(f, P_\varepsilon^{'}) - \varepsilon$ is not an upper bound for $\{L(f,P): \ P\}$. Therefore there exists a partition $P_\varepsilon^{''}$ such that

$$U(f, P_\varepsilon^{'}) - \varepsilon < L(f, P_\varepsilon^{''}).$$

Now let $P_\varepsilon = P_\varepsilon^{'} \cup P_\varepsilon^{''}$ be the common refinement, then by Lemma 5.21 we have

$$U(f, P_\varepsilon) - \varepsilon \leqslant U(f, P_\varepsilon^{'}) - \varepsilon < L(f, P_\varepsilon^{''}) \leqslant L(f, P_\varepsilon).$$

*(continued).* Conversely, suppose $f$ satisfies the condition in the statement but is not Riemann integrable, in other words $L(f) < U(f)$. Let $\varepsilon = U(f) - L(f)$ and let $P_\varepsilon$ be a partition such that

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon.$$

Since $U(f, P_\varepsilon) \geqslant U(f)$ and $L(f, P_\varepsilon) \leqslant L(f)$, we get

$$\varepsilon > U(f, P_\varepsilon) - L(f, P_\varepsilon) \geqslant U(f) - L(f) = \varepsilon,$$

contradiction. □

**Theorem 5.28.** *Let* $f : [a,b] \longrightarrow \mathbf{R}$*. If* $f$ *is continuous on* $[a,b]$*, then it is integrable on* $[a,b]$*.*

We are missing some crucial ingredients for this proof, so we omit it for the time being.

**Theorem 5.29.** *Let $f, g : [a,b] \longrightarrow \mathbf{R}$ be integrable on $[a,b]$, $\lambda \in \mathbf{R}$.*

*(a)* $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ *for all* $c \in [a,b]$.

*(b)* $\displaystyle\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$

*(c)* $\displaystyle\int_a^b \lambda f(x)\,dx = \lambda \int_a^b f(x)\,dx.$

*(d)* $\displaystyle\left|\int_a^b f(x)\,dx\right| \leqslant \int_a^b |f(x)|\,dx.$

*(e) If* $f(x) \leqslant g(x)$ *for all* $x \in [a,b]$, *then* $\displaystyle\int_a^b f(x)\,dx \leqslant \int_a^b g(x)\,dx.$

Here is a proof of part (a). We use Theorem 5.27.

Let $\varepsilon > 0$.

Since $f$ is integrable on $[a,b]$, by Theorem 5.27 there exists a partition $P = P_\varepsilon$ of $[a,b]$ such that

$$U(f,P) - L(f,P) < \varepsilon.$$

Let $P' = P \cup \{c\}$, then $P'$ is a refinement of $P$ and

$$U(f,P') - L(f,P') < \varepsilon.$$

Let $P_1' = P' \cap [a,c]$ and $P_2' = P' \cap [c,b]$, then

$$U(f,P') = U(f,P_1') + U(f,P_2') \quad \text{and} \quad L(f,P') = L(f,P_1') + L(f,P_2').$$

Therefore

$$\big(U(f,P_1') - L(f,P_1')\big) + \big(U(f,P_2') - L(f,P_2')\big) < \varepsilon,$$

but both of these summands are non-negative, so each is $< \varepsilon$. Hence, by Theorem 5.27, $f$ is integrable on $[a,c]$ and on $[c,b]$.

Moreover,

$$\int_a^b f(x)\,dx \leqslant U(f,P') = U(f,P_1') + U(f,P_2') < L(f,P_1') + L(f,P_2') + \varepsilon \leqslant \int_a^c f(x)\,dx + \int_c^b f(x)\,dx + \varepsilon$$

and

$$\int_a^c f(x)\,dx + \int_c^b f(x)\,dx \leqslant U(f,P_1') + U(f,P_2') < L(f,P_1') + L(f,P_2') + \varepsilon = L(f,P') + 2\varepsilon \leqslant \int_a^b f(x)\,dx + \varepsilon.$$

**Theorem 5.30** (Fundamental Theorem of Calculus I)**.** *Let* $f : [a,b] \longrightarrow \mathbf{R}$ *be integrable. Let* $F : [a,b] \longrightarrow \mathbf{R}$ *be differentiable such that* $F'(x) = f(x)$ *for all* $x \in [a,b]$*. Then*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

*Proof.* Let $P = \{x_0, x_1, \ldots, x_n\}$ be a partition of $[a,b]$. Let $i \in \{1, \ldots, n\}$ and consider the interval $[x_{i-1}, x_i]$. By the Mean Value Theorem (Theorem 5.10), there exists $c_i \in (x_{i-1}, x_i)$ such that

$$f(c_i) = F'(c_i) = \frac{F(x_i) - F(x_{i-1})}{x_i - x_{i-1}}.$$

Therefore

$$\begin{gathered} m_i \leqslant f(c_i) \leqslant M_i \\ m_i(x_i - x_{i-1}) \leqslant F(x_i) - F(x_{i-1}) \leqslant M_i(x_i - x_{i-1}). \end{gathered}$$

Summing over all $i \in \{1, \ldots, n\}$ we get

$$L(f,P) = \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \leqslant \sum_{i=1}^{n} \big(F(x_i) - F(x_{i-1})\big) \leqslant \sum_{i=1}^{n} M_i(x_i - x_{i-1}) = U(f,P).$$

Note that the sum in the middle simplifies to $F(b) - F(a)$, hence

$$L(f, P) \leqslant F(b) - F(a) \leqslant U(f, P).$$

The two inequalities hold for all partitions $P$ of $[a, b]$, so

$$L(f) \leqslant F(b) - F(a) \leqslant U(f).$$

But $f$ is integrable on $[a, b]$, so

$$L(f) = U(f) = \sum_{a}^{b} f(x)\, dx \qquad \Rightarrow \qquad F(b) - F(a) = \int_a^b f(x)\, dx. \qquad \square$$

**Theorem 5.31** (Fundamental Theorem of Calculus II)**.** *Let $g\colon [a,b] \longrightarrow \mathbf{R}$ be integrable. Define $G\colon [a,b] \longrightarrow \mathbf{R}$ by*

$$G(x) = \int_a^x g(t)\,dt.$$

*Then*

- *(a) the function $G$ is continuous on $[a,b]$;*
- *(b) if $g$ is continuous at $c \in [a,b]$, then $G$ is differentiable at $c$ and $G'(c) = g(c)$.*

*Proof.* (a) The function $g$ is integrable hence bounded, so there exists $M \in \mathbf{R}$ such that

$$|g(t)| \leqslant M \qquad \text{for all } t \in [a,b].$$

Let $x \leqslant c$ with $x, c \in [a,b]$, then

$$|G(c) - G(x)| = \left|\int_a^c g(t)\,dt - \int_a^x g(t)\,dt\right| = \left|\int_x^c g(t)\,dt\right| \leqslant \int_x^c |g(t)|\,dt \leqslant \int_x^c M\,dt = M(c-x).$$

If instead $x > c$ then we interchange $x$ and $c$ as integration limits; the upshot is that

$$|G(c) - G(x)| \leqslant M|c-x| \qquad \text{for all } x, c \in [a,b].$$

Do you see why this implies that $G$ is continuous at any $c \in [a, b]$?

(b) For any $x \in [a, b]$ we have

$$\frac{G(x) - G(c)}{x - c} = \frac{1}{x - c}\left(\int_a^x g(t)\,dt - \int_a^c g(t)\,dt\right) = \frac{1}{x - c}\int_c^x g(t)\,dt.$$

Let $\varepsilon > 0$. Since $g$ is continuous at $c$, there exists $\delta > 0$ such that for all $t \in [a, b]$, if $|t - c| < \delta$ then $|g(t) - g(c)| < \varepsilon$.

Let $x \in [a, b]$ be such that $0 < |x - c| < \delta$. Suppose $x > c$ (the case $x < c$ is similar). For all $t \in [c, x]$ we have $|g(t) - g(c)| < \varepsilon$, so that

$$\begin{gathered} g(c) - \varepsilon < g(t) < g(c) + \varepsilon \\ \big(g(c) - \varepsilon\big)(x - c) = \int_c^x \big(g(c) - \varepsilon\big)\,dt < \int_c^x g(t)\,dt < \int_c^x \big(g(c) + \varepsilon\big)\,dt = \big(g(c) + \varepsilon\big)(x - c) \\ g(c) - \varepsilon < \frac{1}{x - c}\int_c^x g(t)\,dt < g(c) + \varepsilon, \end{gathered}$$

meaning that

$$\left|\frac{G(x) - G(c)}{x - c} - g(c)\right| = \left|\left(\frac{1}{x - c}\int_c^x g(t)\,dt\right) - g(c)\right| < \varepsilon,$$

hence $G'(c)$ exists and equals $g(c)$. □

**Example 5.32.** Fix $k \in \mathbf{R}$ and let $g : [a, b] \longrightarrow \mathbf{R}$ be the constant function $g(x) = k$.

We have

$$G(x) = \int_a^x k \, dt = k(x - a) \qquad \text{for all } x \in [a, b].$$

This is differentiable at every $c \in [a, b]$ with derivative $G'(c) = k$.