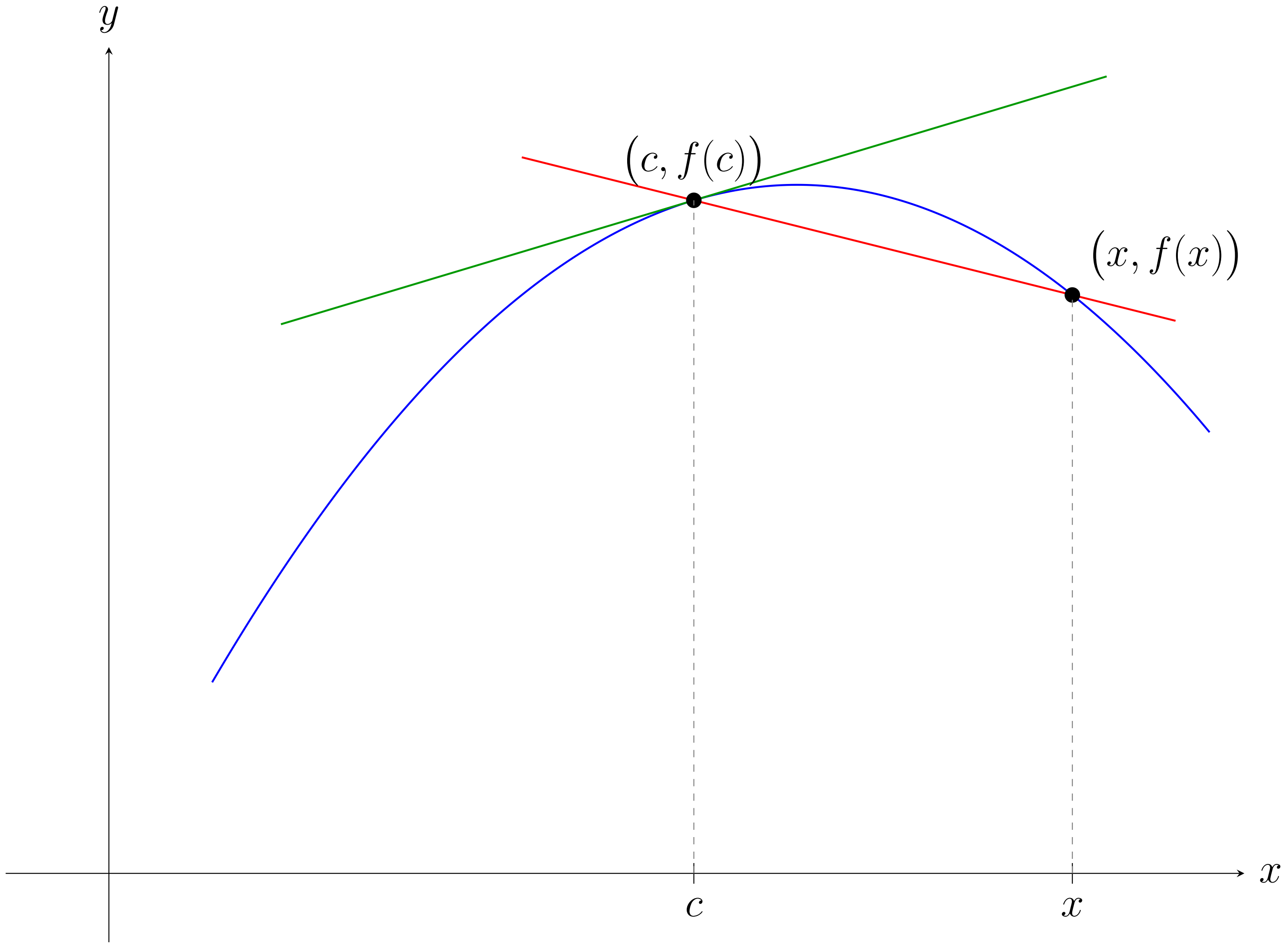
5 Differentiation and integration

Chapter contents

5.1 Derivatives



The geometric definition of the derivative of f at c is: the slope of the tangent line to the curve $y = f(x)$ at the point $\big(c, f(c)\big)$.

Definition 5.1. Let $I \subseteq \mathbf{R}$ be an interval, $f : I \longrightarrow \mathbf{R}$ a function, $c \in I$.

We say that f is *differentiable at c* if there exists $L_c \in \mathbf{R}$ such that

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = L_c.$$

The number L_c is called the *derivative of f at c*.

If f is differentiable at c for all $c \in I$, we say that f is *differentiable on I*; the *derivative of f* is the function $f' : I \longrightarrow \mathbf{R}$ defined by $f'(c) = L_c$.

Example 5.2. The function $f : \mathbf{R} \longrightarrow \mathbf{R}$ given by $f(x) = x^2$ is differentiable on $\mathbf{R}$.

Example 5.3. The function $g : \mathbf{R} \longrightarrow \mathbf{R}$ given by $g(x) = |x|$ is not differentiable at 0.

Theorem 5.4. *Let* $I \subseteq \mathbf{R}$ *be an interval,* $f: I \longrightarrow \mathbf{R}$ *a function,* $c \in I$. *If* f *is differentiable at* c, *then* f *is continuous at* c.

Theorem 5.5 (Linearity)**.** *Let $I \subseteq \mathbf{R}$ be an interval, $c \in I$, $f, g\colon\ I \longrightarrow \mathbf{R}$ two functions that are differentiable at c. Then*

- *$f+g\colon\ I \longrightarrow \mathbf{R}$ is differentiable at c and $(f+g)'(c) = f'(c) + g'(c)$;*
- *for any $\lambda \in \mathbf{R}$, $\lambda f\colon\ I \longrightarrow \mathbf{R}$ is differentiable at c and $(\lambda f)'(c) = \lambda f'(c)$.*

Let Fun be the real vector space of all functions $\mathbf{R} \longrightarrow \mathbf{R}$; let Diff_I be the subset of functions that are differentiable on I.

The Theorem says that

Theorem 5.6 (Product Rule)**.** *Let* $I \subseteq \mathbf{R}$ *be an interval,* $c \in I$*,* $f, g\colon\ I \longrightarrow \mathbf{R}$ *two functions that are differentiable at* c*. Then the product* $fg\colon\ I \longrightarrow \mathbf{R}$ *is differentiable at* c *and*

$$(fg)'(c) = f'(c)g(c) + f(c)g'(c).$$

Example 5.7. For each $n \in \mathbf{Z}_{\geqslant 1}$ the function $f : \mathbf{R} \longrightarrow \mathbf{R}$ given by $f(x) = x^n$ is differentiable on $\mathbf{R}$ with derivative nx^{n-1}.

One of the big wins of derivatives is that they can be used to optimise functions.

Definition 5.8. Let $f : [a, b] \longrightarrow \mathbf{R}$ be a function, $c \in (a, b)$.

- We say that f has a *local maximum* at c if there exists $\delta > 0$ such that

$$\text{for all } x \in [a, b], \text{ if } |x - c| < \delta \text{ then } f(x) \leqslant f(c).$$

- We say that f has a *local minimum* at c if there exists $\delta > 0$ such that

$$\text{for all } x \in [a, b], \text{ if } |x - c| < \delta \text{ then } f(x) \geqslant f(c).$$

Recall that if f is continuous on $[a,b]$, it attains its extremal values on $[a,b]$ (by the Extremal Value Theorem).

Theorem 5.9. *Let* $f : [a,b] \longrightarrow \mathbf{R}$ *be continuous, and suppose that* f *is differentiable at some* $c \in (a,b)$*. If* f *has a local maximum or minimum at* c*, then* $f'(c) = 0$*.*

If we know that f is differentiable on $[a, b]$, we can say something about the values of the derivative f' on the interval $[a, b]$:

Theorem 5.10 (Mean Value Theorem)**.** *Let* $a < b$, $f : [a, b] \longrightarrow \mathbf{R}$ *be continuous on* $[a, b]$ *and differentiable on* (a, b). *There exists* $c \in (a, b)$ *such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

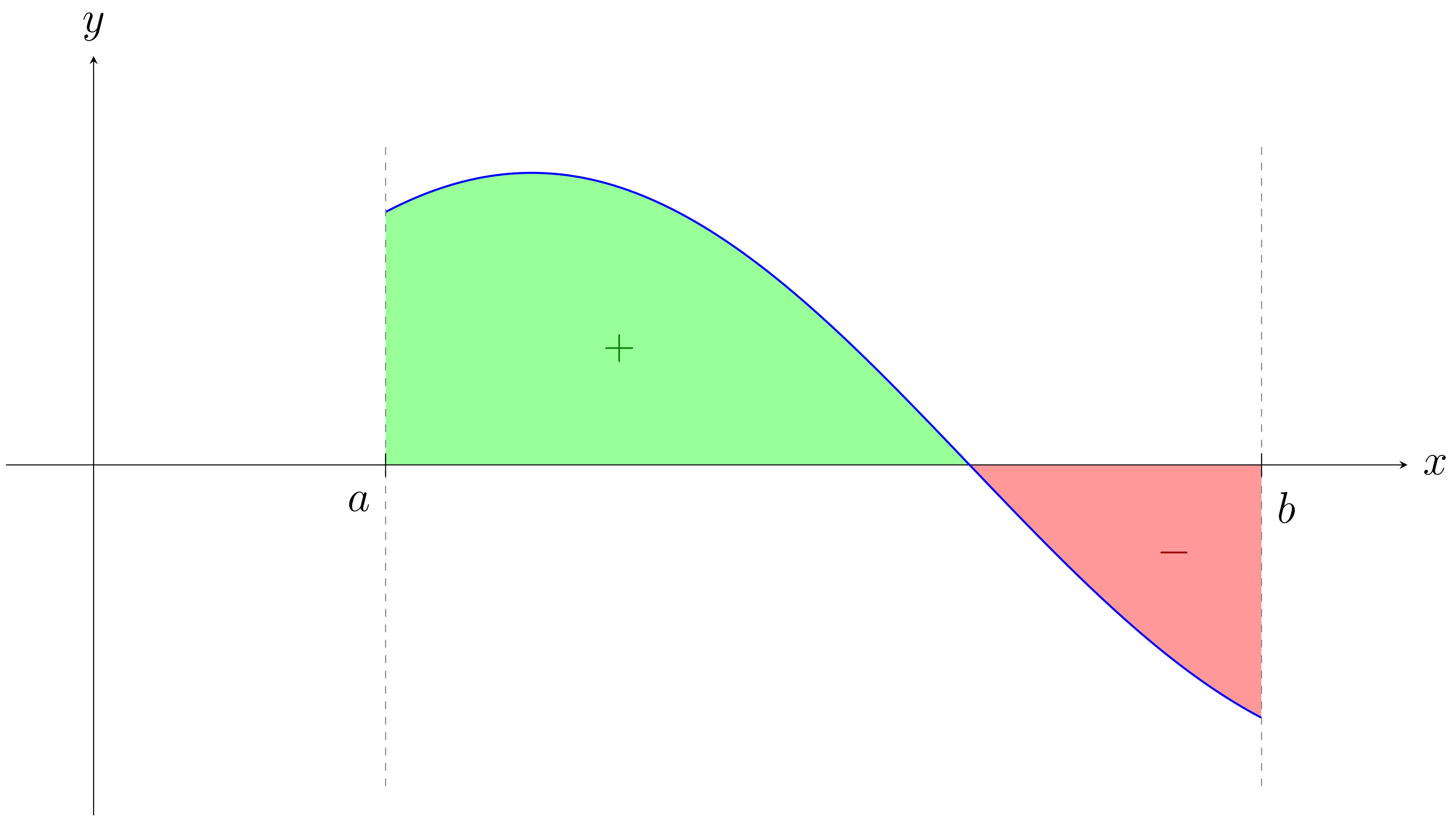
Example 5.11. If you drive 6 km in 5 minutes on a road with a speed limit of 60 km/h, should you get a speeding ticket?

We start by proving a special case of the Mean Value Theorem:

Theorem 5.12 (Rolle)**.** *Let* $a < b$*,* $f : [a, b] \longrightarrow \mathbf{R}$ *be continuous on* $[a, b]$ *and differentiable on* (a, b)*. If* $f(a) = f(b) = 0$*, then there exists* $c \in (a, b)$ *such that* $f'(c) = 0$*.*

Now we can prove the Mean Value Theorem:

5.2 Integrals



The geometric definition of the definite integral of a function f on a closed interval $[a, b]$ of finite length is the area of the region between the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$, where areas above the x-axis are taken with a positive sign, and areas below the x-axis with a negative sign.

Definition 5.13. A *partition* of the interval $[a, b]$ is a set

$$P = \{x_0, x_1, \ldots, x_{n-1}, x_n\},$$

where $n \in \mathbf{Z}_{\geqslant 1}$ and

$$a = x_0 \leqslant x_1 \leqslant x_2 \leqslant \ldots \leqslant x_{n-1} \leqslant x_n = b.$$

The intervals $[x_{i-1}, x_i]$ are called *subintervals* of $[a, b]$.

Example 5.14.

Definition 5.15. Let $f: [a,b] \longrightarrow \mathbf{R}$ be bounded on $[a,b]$ and let $P = \{x_0, \dots, x_n\}$ be a partition of $[a,b]$.

The *lower Riemann sum* of f with respect to P is

$$L(f,P) = \sum_{k=1}^{n} m_k(x_k - x_{k-1}) = m_1(x_1 - x_0) + m_2(x_2 - x_1) + \cdots + m_n(x_n - x_{n-1}),$$

where for $k = 1, \dots, n$ we have

$$m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\}.$$

The *upper Riemann sum* of f with respect to P is

Example 5.16. Find lower and upper bounds for the area under $f:\ [1,4] \longrightarrow \mathbf{R}$, $f(x) = 25 - x^2$, for $1 \leqslant x \leqslant 4$.

Lemma 5.17. *Let $f : [a,b] \longrightarrow \mathbf{R}$ be bounded on $[a,b]$. For every partition P of $[a,b]$ we have*

$$L(f,P) \leqslant U(f,P).$$

It feels like more should be true though: it should be the case that any lower sum (with respect to any partition) should be less than any upper sum (with respect to any other partition). This takes a bit more work.

Definition 5.18. If P and Q are partitions of $[a, b]$, we say that Q is a *refinement* of P if $P \subseteq Q$.

Example 5.19. For the interval $[1, 5]$, $Q = \{1, 2.5, 3, \pi, 4, 4.1, 5\}$ is a refinement of the partition $P = \{1, 3, 4, 5\}$.

Lemma 5.20. *Let $f : [a, b] \longrightarrow \mathbf{R}$ be bounded on $[a, b]$, R a partition of $[a, b]$, and Q a refinement of R such that $Q \setminus R$ has a single element. Then*

$$L(f, R) \leqslant L(f, Q) \qquad \textit{and} \qquad U(f, Q) \leqslant U(f, R).$$

Lemma 5.21. *Let* $f : [a, b] \longrightarrow \mathbf{R}$ *be bounded on* $[a, b]$*,* P *a partition of* $[a, b]$*, and* Q *a refinement of* P*. Then*

$$L(f, P) \leqslant L(f, Q) \qquad \textit{and} \qquad U(f, Q) \leqslant U(f, P).$$

Theorem 5.22. *Let* $f : [a, b] \longrightarrow \mathbf{R}$ *be bounded on* $[a, b]$ *and* P, P' *partitions of* $[a, b]$*. Then*

$$L(f, P) \leqslant U(f, P').$$

Definition 5.23. Let $f : [a, b] \longrightarrow \mathbf{R}$ be bounded on $[a, b]$.

The *lower Riemann integral* of f on $[a, b]$ is

$$L(f) = \sup \{L(f, P) : \ P \text{ is a partition of } [a, b]\}.$$

The *upper Riemann integral* of f on $[a, b]$ is

$$U(f) = \inf \{U(f, P) : \ P \text{ is a partition of } [a, b]\}.$$

Theorem 5.24. *Let $f : [a, b] \longrightarrow \mathbf{R}$ be bounded on $[a, b]$. Then*

$$L(f) \leqslant U(f).$$

But surely we actually have equality $L(f) = U(f)$?

Example 5.25. Consider the function $f : [1, 2] \longrightarrow \mathbf{R}$ given by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbf{Q}, \\ 0 & \text{if } x \notin \mathbf{Q}. \end{cases}$$

Definition 5.26. Let $f : [a,b] \longrightarrow \mathbf{R}$ be bounded on $[a,b]$.

We say that f is *Riemann integrable* on $[a,b]$ if $L(f) = U(f)$, in which case we define the *Riemann integral* of f on $[a,b]$ to be

$$\int_a^b f(x)\,dx = L(f) = U(f).$$

Example 5.25 exhibited a non-integrable function.

Theorem 5.27. *Let* $f : [a, b] \longrightarrow \mathbf{R}$ *be bounded on* $[a, b]$*. The function* f *is Riemann integrable on* $[a, b]$ *if and only if for every* $\varepsilon > 0$ *there exists a partition* P_ε *of* $[a, b]$ *such that*

$$U(f, P_\varepsilon) - L(f, P_\varepsilon) < \varepsilon.$$

Theorem 5.28. *Let* $f : [a, b] \longrightarrow \mathbf{R}$. *If* f *is continuous on* $[a, b]$, *then it is integrable on* $[a, b]$.

Theorem 5.29. *Let* $f, g : [a,b] \longrightarrow \mathbf{R}$ *be integrable on* $[a,b]$, $\lambda \in \mathbf{R}$.

(a) $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ *for all* $c \in [a,b]$.

(b) $\displaystyle\int_a^b (f(x)+g(x))\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.

(c) $\displaystyle\int_a^b \lambda f(x)\,dx = \lambda \int_a^b f(x)\,dx$.

(d) $\displaystyle\left|\int_a^b f(x)\,dx\right| \leqslant \int_a^b |f(x)|\,dx$.

(e) If $f(x) \leqslant g(x)$ *for all* $x \in [a,b]$, *then* $\displaystyle\int_a^b f(x)\,dx \leqslant \int_a^b g(x)\,dx$.

Theorem 5.30 (Fundamental Theorem of Calculus I)**.** *Let* $f : [a, b] \longrightarrow \mathbf{R}$ *be integrable. Let* $F : [a, b] \longrightarrow \mathbf{R}$ *be differentiable such that* $F'(x) = f(x)$ *for all* $x \in [a, b]$*. Then*

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Theorem 5.31 (Fundamental Theorem of Calculus II)**.** *Let* $g : [a, b] \longrightarrow \mathbf{R}$ *be integrable. Define* $G : [a, b] \longrightarrow \mathbf{R}$ *by*

$$G(x) = \int_a^x g(t)\,dt.$$

Then

(a) the function G *is continuous on* $[a, b]$*;*

(b) if g *is continuous at* $c \in [a, b]$*, then* G *is differentiable at* c *and* $G'(c) = g(c)$*.*

Example 5.32. Fix $k \in \mathbf{R}$ and let $g: [a, b] \longrightarrow \mathbf{R}$ be the constant function $g(x) = k$.