4 Functional limits and continuity

Chapter contents

4.1 Limits of functions

We shift our attention from limits of sequences (that is, functions $\mathbf{N} \longrightarrow \mathbf{R}$) as the variable n tends to ∞, to limits of functions $\mathbf{R} \longrightarrow \mathbf{R}$ as the variable x tends to some point $a \in \mathbf{R}$ (or, a little later, to ∞).

For instance, you may have seen that for the function $f: \mathbf{R} \longrightarrow \mathbf{R}$ defined by $f(x) = 2x$, we have that $f(x)$ tends to 6 as $x \to 3$:

x	$f(x) = 2x$
4	8
3.5	7
3.1	6.2
3.01	6.02
3.001	6.002
etc.	

Easy, right?

Almost.

The definition of limit of a function should be flexible enough to cover functions that are much less well-behaved than $f(x) = 2x$. We will consider some of the issues that can arise.

Example 4.1. Let $f : \mathbf{R} \longrightarrow \mathbf{R}$ be given by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbf{Q}, \\ -1 & \text{otherwise.} \end{cases}$$

x	$f(x)$	x	$f(x)$
1	1	$\sqrt{2} - 0.1$	-1
1.4	1	$\sqrt{2} - 0.01$	-1
1.41	1	$\sqrt{2} - 0.001$	-1
1.414	1	$\sqrt{2} - 0.0001$	-1
1.4142	1	$\sqrt{2} - 0.00001$	-1

So ... is the function approaching 1, or -1, as $x \longrightarrow \sqrt{2}$?

Remember the sequence $\left((-1)^n\right)$ from the previous chapter?

On the other hand, jumps in the graph are **not necessarily** a dealbreaker:

Example 4.2. Let $f : \mathbf{R} \longrightarrow \mathbf{R}$ be given by

$$f(x) = \begin{cases} 1 & \text{if } x \neq 3, \\ -1 & \text{otherwise.} \end{cases}$$

There is a jump in the graph at $x = 3$. What can we say about $f(x)$ as $x \longrightarrow 3$?

How about for $g : \mathbf{R} \longrightarrow \mathbf{R}$ given by

$$g(x) = \begin{cases} 1 & \text{if } x < 3, \\ -1 & \text{otherwise?} \end{cases}$$

Many functions are not defined on all of $\mathbf{R}$ e.g. $f:\ \mathbf{R}_{\geqslant 0} \longrightarrow \mathbf{R}$ given by $f(x)=\sqrt{x}$ or

$g:\ \mathbf{R}\setminus\{0\} \longrightarrow \mathbf{R}$ given by $g(x)=1/x$.

Let $E \subseteq \mathbf{R}$ be an arbitrary subset. At what points a would it make sense to talk about limits for functions $f:\ E \longrightarrow \mathbf{R}$?

We cannot just take any $a \in \mathbf{R}$:

it doesn't make sense to ask about the behaviour of $f:\ \mathbf{R}_{\geqslant 0} \longrightarrow \mathbf{R}$, $f(x)=\sqrt{x}$, as $x \longrightarrow -5$.

On the other hand, restricting to $a \in E$ is not quite right either:

for $g:\ E=\mathbf{R}\setminus\{2\} \longrightarrow \mathbf{R}$ given by

$$g(x)=\frac{x^2-4}{x-2},$$

it makes sense to ask about $x \longrightarrow 2$, but $2 \notin E$.

Limit points of sets

Definition 4.3. Let $E \subseteq \mathbf{R}$. An element $a \in \mathbf{R}$ is a *limit point of E* if for every $\delta > 0$ there exists $x \in E$ such that

$$0 < |x - a| < \delta.$$

Example 4.4. Consider the function $f : (0, \infty) \longrightarrow \mathbf{R}$ given by $f(x) = \log(x)$.

The domain of f is $E = (0, \infty)$.

The set of limit points of E is $[0, \infty)$. In other words, $a \in \mathbf{R}$ is a limit point of E if and only if $a \geqslant 0$.

Suppose $a \geqslant 0$. Let $\delta > 0$. Take $x = a + \frac{\delta}{2}$, then

$$a - \delta < a < x = a + \frac{\delta}{2} < a + \delta,$$

so $0 < |x - a| < \delta$ and a is a limit point of E.

Suppose $a < 0$. Take $\delta = |a|$. Suppose $x \in \mathbf{R}$ is such that $0 < |x - a| < \delta$, then $x < a + \delta = a - a = 0$, so $x \notin E$. Hence a is not a limit point of E.

Example 4.5. What is the set of limit points of $E = \mathbf{N} \subseteq \mathbf{R}$?

It is the empty set.

If $a \in \mathbf{N}$, take $\delta = 1$, then the only $x \in \mathbf{N}$ with $|x - a| < 1$ is $x = a$. Hence a is not a limit point of $\mathbf{N}$.

In Tutorial Question 7.3 you will show that if $a \in \mathbf{R} \setminus \mathbf{N}$ then a is not a limit point of $\mathbf{N}$.

For more examples of limit points of sets, see Exercise 3.20 and Tutorial Question 7.4.

Theorem 4.6. *Let $E \subseteq \mathbf{R}$ and $a \in \mathbf{R}$. Then a is a limit point of E if and only if there exists a sequence (x_n) such that $x_n \longrightarrow a$, and for every $n \in \mathbf{N}$ we have $x_n \in E$ and $x_n \neq a$.*

This is Exercise 3.21.

Limit of a function

Definition 4.7. Let $E \subseteq \mathbf{R}$, let a be a limit point of E, let $f : E \longrightarrow \mathbf{R}$ be a function, and let $L \in \mathbf{R}$.

We say that L is the *limit of f as $x \longrightarrow a$* and write

$$\lim_{x \to a} f(x) = L \qquad \text{or} \qquad f(x) \longrightarrow L \text{ as } x \longrightarrow a,$$

if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in E$ and $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.

Example 4.8. For any $a, c \in \mathbf{R}$ we have

$$\lim_{x \to a} x = a \qquad \text{and} \qquad \lim_{x \to a} c = c.$$

Let $\varepsilon > 0$. Take $\delta = \varepsilon$. If $x \in \mathbf{R}$ and $0 < |x - a| < \delta$, then $|x - a| < \varepsilon$.

Let $\varepsilon > 0$. Take $\delta = 1$ (or whatever other positive number you like). If $x \in \mathbf{R}$ and $0 < |x - a| < \delta$, then $|c - c| < \varepsilon$.

Example 4.9.

$$\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = 4.$$

Scrap work. Fix $\varepsilon > 0$ and "solve for" δ:

$$\begin{aligned} \left| \frac{x^2 - 4}{x - 2} - 4 \right| &< \varepsilon \\ \left| \frac{(x-2)(x+2)}{x - 2} - 4 \right| &< \varepsilon \\ |(x+2) - 4| &< \varepsilon \\ |x - 2| &< \varepsilon. \end{aligned}$$

So: take $\delta = \varepsilon$.

Proof. Let $\varepsilon > 0$. Take $\delta = \varepsilon$.

Let $x \in \mathbf{R}$ be such that $0 < |x - 2| < \delta$. Then:

$$\begin{aligned} |x - 2| &< \varepsilon \\ |(x+2) - 4| &< \varepsilon \\ \left| \frac{(x-2)(x+2)}{(x - 2)} - 4 \right| &< \varepsilon \\ \left| \frac{x^2 - 4}{x - 2} - 4 \right| &< \varepsilon. \end{aligned}$$

We conclude that $\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = 4$.

Theorem 4.10 (Sequential Criterion for Function Limits)**.** *Let* $E \subseteq \mathbf{R}$*,* $a \in \mathbf{R}$ *a limit point of* E*,* $f : E \longrightarrow \mathbf{R}$*, and* $L \in \mathbf{R}$*. Then*

$$\lim_{x \to a} f(x) = L$$

if and only if:

$$(*) \quad \textit{for every sequence } (a_n) \textit{ such that } a_n \in E \setminus \{a\} \textit{ for all } n \in \mathbf{N} \textit{ and } a_n \longrightarrow a, \textit{ we have that } f(a_n) \longrightarrow L.$$

Proof. In one direction, suppose $\lim_{x \to a} f(x) = L$ and let (a_n) be a sequence in $E \setminus \{a\}$ such that $a_n \longrightarrow a$. Let $\varepsilon > 0$. There exists $\delta > 0$ such that if $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$. Fix such $\delta > 0$, then there exists $M \in \mathbf{N}$ such that if $n > M$ then $|a_n - a| < \delta$. As $a_n \in E \setminus \{a\}$, we have $0 < |a_n - a| < \delta$ for all $n > M$, therefore $|f(a_n) - L| < \varepsilon$ for all $n > M$. We conclude that $f(a_n) \longrightarrow L$.

Conversely, suppose (*) holds. We proceed by contradiction to prove that $\lim_{x \to a} f(x) = L$. Suppose this is not the case, then there exists $\varepsilon > 0$ such that for all $\delta > 0$ there exists $x \in E$ with $0 < |x - a| < \delta$ and $|f(x) - L| \geqslant \varepsilon$. In particular, for each $n \in \mathbf{N}$ with $n \geqslant 1$, take $\delta = 1/n$ and let a_n be the corresponding x, so that $a_n \in E$ with $0 < |a_n - a| < 1/n$ and $|f(a_n) - L| \geqslant \varepsilon$. Then the sequence $(f(a_n))$ does not converge to L, but for all $n \geqslant 1$ we have $a_n \in E \setminus \{a\}$ and $a - 1/n < a_n < a + 1/n$, so by the Sandwich Theorem $a_n \longrightarrow a$, contradicting (*). □

This result gives us access to a lot of the theorems from the chapter on sequences!

Theorem 4.11 (Algebra of Limits). *Let $E \subseteq \mathbf{R}$, $a \in \mathbf{R}$ a limit point of E, and $f, g : E \longrightarrow \mathbf{R}$ such that*

$$\lim_{x \to a} f(x) = \alpha \quad \text{and} \quad \lim_{x \to a} g(x) = \beta.$$

Then

(a) $\lim_{x \to a} f(x) + g(x) = \alpha + \beta$;

(b) $\lim_{x \to a} f(x)g(x) = \alpha\beta$;

(c) $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\alpha}{\beta}$ *if* $\beta \neq 0$.

Proof. We illustrate the approach by proving part (c). The other parts work the same way.

Let (a_n) be a sequence in $E \setminus \{a\}$ such that $a_n \longrightarrow a$. By the Sequential Criterion for Function Limits, we have $f(a_n) \longrightarrow \alpha$ and $g(a_n) \longrightarrow \beta$. Since $\beta \neq 0$, by the Algebra of Limits for sequences, we have $f(a_n)/g(a_n) \longrightarrow \alpha/\beta$. Therefore, by the Sequential Criterion for Function Limits, we have $\lim_{x \to a} f(x)/g(x) = \alpha/\beta$. □

Theorem 4.12 (Inequalities and Limits)**.** *Let $E \subseteq \mathbf{R}$, $a \in \mathbf{R}$ a limit point of E, and $f, g : E \longrightarrow \mathbf{R}$ such that*

$$\lim_{x\to a} f(x) = \alpha \quad \textit{and} \quad \lim_{x\to a} g(x) = \beta.$$

If $f(x) \leqslant g(x)$ for all $x \in E$, then $\alpha \leqslant \beta$.

For the proof see Tutorial Question 7.7.

Corollary 4.13. *With the same assumptions as in the Theorem:*

If $b \in \mathbf{R}$ is such that $b \leqslant f(x)$ for all $x \in E$, then $b \leqslant \alpha$.

If $c \in \mathbf{R}$ is such that $f(x) \leqslant c$ for all $x \in E$, then $\alpha \leqslant c$.

Example 4.14.

$$\lim_{x\to 1} \frac{x^2-1}{2x-1} = 0.$$

We use the Algebra of Limits several times, as well as Example 4.8.

$$\lim_{x\to 1}(x^2-1) = \left(\lim_{x\to 1} x\right)^2 - 1 = 0$$

$$\lim_{x\to 1}(2x-1) = 2\left(\lim_{x\to 1} x\right) - 1 = 2-1 = 1$$

$$\lim_{x\to 1} \frac{x^2-1}{2x-1} = \frac{0}{1} = 0.$$

Computing this limit directly from the definition is considerably harder!

Infinite limits and limits at infinity

Definition 4.15. Let $E \subseteq \mathbf{R}$, $a \in \mathbf{R}$ a limit point of E, and $f : E \longrightarrow \mathbf{R}$ a function. We write

$$\lim_{x \to a} f(x) = \infty$$

if for every $r \in \mathbf{R}$ there exists $\delta > 0$ such that if $x \in E$ and $0 < |x - a| < \delta$ then $f(x) > r$.

You should write down the analogous definition of $\lim_{x \to a} f(x) = -\infty$.

Example 4.16.

$$\lim_{x \to 0} \frac{1}{x^2} = \infty.$$

We have $f : E \longrightarrow \mathbf{R}$, where $E = \mathbf{R} \setminus \{0\}$.

Let $r \in \mathbf{R}$. If $r \leqslant 0$, we are done, since for every $x \in E$ we have $f(x) = \frac{1}{x^2} > 0 \geqslant r$.

So suppose $r > 0$. Take $\delta = 1/\sqrt{r}$. If $x \in E$ and $0 < |x| < \delta$, then $f(x) = 1/x^2 > 1/\delta^2 = r$.

Definition 4.17. Let $E \subseteq \mathbf{R}$ be a set that is not bounded above, let $f : E \longrightarrow \mathbf{R}$ be a function, and let $L \in \mathbf{R}$. We write

$$\lim_{x \to \infty} f(x) = L$$

if for every $\varepsilon > 0$ there exists $M \in \mathbf{R}$ such that if $x \in E$ and $x > M$ then $|f(x) - L| < \varepsilon$.

You should write down the analogous definition of $\lim_{x \to -\infty} f(x) = L$.

Example 4.18.

$$\lim_{x \to \infty} \frac{1}{x^3} = 0.$$

We have $f: \ E \longrightarrow \mathbf{R}$, where $E = \mathbf{R} \setminus \{0\}$.

Let $\varepsilon > 0$. Take $M = 1/\varepsilon^{1/3}$. For $x \in E$ such that $x > M$ we have

$$|f(x) - 0| = \frac{1}{x^3} < \frac{1}{M^3} < \varepsilon.$$

4.2 Continuity

Definition 4.19. Let $E \subseteq \mathbf{R}$, $f : E \longrightarrow \mathbf{R}$, and $a \in E$. We say that f is *continuous at a* if: for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in E$ and $|x - a| < \delta$ then $|f(x) - f(a)| < \varepsilon$.

We say that f is *continuous on E* (or simply: *continuous*) if it is continuous at every point $a \in E$.

Example 4.20. Show that $f : \mathbf{R} \longrightarrow \mathbf{R}$ given by $f(x) = x$ is continuous.

Let $a \in \mathbf{R}$. Let $\varepsilon > 0$. Take $\delta = \varepsilon$. If $x \in \mathbf{R}$ and $|x - a| < \delta$ then $|f(x) - f(a)| = |x - a| < \delta = \varepsilon$.

You will prove that for any $c \in \mathbf{R}$, the constant function $f : \mathbf{R} \longrightarrow \mathbf{R}$ given by $f(x) = c$ for all $x \in \mathbf{R}$, is continuous, see Tutorial Question 8.1.

Example 4.21. Consider $f:\ \mathbf{R} \longrightarrow \mathbf{R}$ given by $f(x) = x^2$, with $a = 2$.

Let $\varepsilon > 0$. WLOG $\varepsilon < 5$.

Take $\delta = \sqrt{4+\varepsilon} - 2$, then $1 > \delta > 0$ and we have $\delta^2 + 4\delta = \varepsilon$.

If $x \in \mathbf{R}$ is such that $|x - 2| < \delta$, then $x > 0$, so $|x + 2| = x + 2 < 4 + \delta$. Therefore

$$|f(x) - f(2)| = |x^2 - 4| = |x - 2| \cdot |x + 2| < \delta(4 + \delta) = \varepsilon.$$

We conclude that f is continuous at 2.

Let $E \subseteq \mathbf{R}$, $f: E \longrightarrow \mathbf{R}$, and $a \in E$. We have that f is **not** continuous at a if

$$(\exists \varepsilon > 0)(\forall \delta > 0)(\exists x \in E)|x - a| < \delta \text{ and } |f(x) - f(a)| \geqslant \varepsilon.$$

Example 4.22. Consider the following function $f: \mathbf{R} \longrightarrow \mathbf{R}$ at $a = 0$:

$$f(x) = \begin{cases} 2x + 1 & \text{if } x \neq 0, \\ 5 & \text{if } x = 0. \end{cases}$$

Take $\varepsilon = 1$. Let $\delta > 0$ be arbitrary. Take $x = -\frac{\delta}{2}$. Then

$$|x - 0| = |x| = \frac{\delta}{2} < \delta$$

and

$$|f(x) - f(0)| = |(2x + 1) - 5| = |-\delta - 4| = \delta + 4 > 4 \geqslant 1.$$

Theorem 4.23 (Limit Criterion for Continuity)**.** *Let $E \subseteq \mathbf{R}$, $f : E \longrightarrow \mathbf{R}$, and $a \in E$ be a limit point of E. The function f is continuous at a if and only if $\lim_{x \to a} f(x) = f(a)$.*

Proof. Suppose f is continuous at a. Let $\varepsilon > 0$. There exists $\delta > 0$ such that if $x \in E$ and $|x - a| < \delta$ then $|f(x) - f(a)| < \varepsilon$. In particular, if $x \in E$ and $0 < |x - a| < \delta$ then $|f(x) - f(a)| < \varepsilon$, therefore $f(x) \longrightarrow f(a)$ as $x \longrightarrow a$.

Conversely, suppose $f(x) \longrightarrow f(a)$ as $x \longrightarrow a$. Let $\varepsilon > 0$. There exists $\delta > 0$ such that if $x \in E$ and $0 < |x - a| < \delta$ then $|f(x) - f(a)| < \varepsilon$. If $x = a$ then $|f(x) - f(a)| = |f(a) - f(a)| = 0 < \varepsilon$. Therefore if $x \in E$ and $|x - a| < \delta$ then $|f(x) - f(a)| < \varepsilon$, hence f is continuous at a. □

You will prove that if $a \in E$ is **not** a limit point of E, then any function $f : E \longrightarrow \mathbf{R}$ is continuous at a, see Tutorial Question 8.3.

Theorem 4.24 (Algebra of Continuity)**.** *Let $E \subseteq \mathbf{R}$, $f, g : E \longrightarrow \mathbf{R}$, $a \in E$. If f and g are both continuous at a, then*

(a) $f + g$ is continuous at a;

(b) fg is continuous at a;

(c) f/g is continuous at a if $g(a) \neq 0$.

Follows immediately from the Limit Criterion for Continuity and from the Algebra of Limits for functions.

Corollary 4.25. *All polynomial functions* $\mathbf{R} \longrightarrow \mathbf{R}$ *are continuous.*

Proof. Slight (unimportant) subtlety: the degree of a nonzero constant polynomial c_0 is definitely 0. But the degree of the zero polynomial 0 is more controversial. There are some good reasons to make it be $-\infty$. Whichever side of this you are on, we know that all constant functions are continuous, in particular the zero polynomial is continuous.

So we now focus on non-zero polynomials, whose degree $n \in \mathbf{N}$. We proceed by induction.

Base case $n = 0$: $f(x) = c_0 \neq 0$ is a constant function, hence continuous by Tutorial Question 8.1.

Induction step: let $k \in \mathbf{N}$ be arbitrary but fixed and suppose that all polynomials of degree k are continuous. Let f be an arbitrary polynomial of degree $k+1$, then

$$f(x) = c_0 + c_1 x + \cdots + c_{k+1} x^{k+1} = c_0 + x\big(c_1 + c_2 x + \cdots + c_{k+1} x^k\big) = c_0 + x g(x),$$

where g is a polynomial of degree k. By the induction hypothesis, g is continuous, and so is x, so the product $xg(x)$ is continuous by the Algebra of Continuity. The constant function c_0 is continuous and $xg(x)$ is continuous, so $x_0 + xg(x) = f(x)$ is continuous by the Algebra of Continuity. □

Once these functions are properly defined, one can show that they are continuous (on their respective domains of definition): e^x, $\sin(x)$, $\cos(x)$, $\log(x)$, $x^{1/p}$ for $p \in \mathbf{N}_{\geqslant 1}$.

Theorem 4.26 (Sequential Criterion for Continuity)**.** *Let $E \subseteq \mathbf{R}$, $f: E \longrightarrow \mathbf{R}$, $a \in E$. The function f is continuous at a if and only if:*

$$(*) \quad \textit{for every sequence } (a_n) \textit{ in } E \textit{ such that } a_n \longrightarrow a \textit{ we have } f(a_n) \longrightarrow f(a).$$

Proof. Suppose f is continuous at a and let (a_n) be a sequence in E that converges to a. Let $\varepsilon > 0$. There exists $\delta > 0$ such that if $x \in E$ such that $|x - a| < \delta$ then $|f(x) - f(a)| < \varepsilon$. Given this $\delta > 0$, there exists $M \in \mathbf{N}$ such that if $n > M$ then $|a_n - a| < \delta$, so $|f(a_n) - f(a)| < \varepsilon$. Therefore $f(a_n) \longrightarrow f(a)$.

Conversely, suppose f satisfies (*). If $a \in E$ is not a limit point of E, then f is automatically continuous at a, see Tutorial Question 8.3. If $a \in E$ is a limit point of E, then by the Sequential Criterion for Function Limits we get that $\lim_{x \to a} f(x) = f(a)$, hence by the Limit Criterion for Continuity we conclude that f is continuous at a. □

Theorem 4.27 (Composition and Continuity)**.** *Let $A, B \subseteq \mathbf{R}$, $f : A \longrightarrow \mathbf{R}$ and $g : B \longrightarrow \mathbf{R}$ such that $f(A) \subseteq B$. Let $a \in A$. If f is continuous at a and g is continuous at $f(a)$, then $g \circ f$ is continuous at a.*

Proof. We use the Sequential Criterion for Continuity. Let (a_n) be a sequence in A such that $a_n \longrightarrow a$. Since f is continuous at a, we have $f(a_n) \longrightarrow f(a)$. Therefore $\big(f(a_n)\big)$ is a sequence in B such that $f(a_n) \longrightarrow f(a)$; since g is continuous at $f(a)$, we have that $g\big(f(a_n)\big) \longrightarrow g\big(f(a)\big)$. In other words, $(g \circ f)(a_n) \longrightarrow (g \circ f)(a)$. Using the Sequential Criterion for Continuity once more, we conclude that $g \circ f$ is continuous at a. □

In Exercise 4.10 you are asked to write a proof of the Composition and Continuity Theorem that uses the $\varepsilon - \delta$ definition directly.

Let $E \subseteq \mathbf{R}$, $f: E \longrightarrow \mathbf{R}$.

For a subset $A \subseteq E$, let $f|_A: A \longrightarrow \mathbf{R}$ denote the *restriction of f to A*:

$$f|_A(x) = f(x) \qquad \text{for all } x \in A.$$

You will show that if $a \in A$ and f is continuous at a, then $f|_A$ is continuous at a, see Tutorial Question 8.4.

Although it may seem fairly trivial, restriction is a useful thing. For instance, the function $f: \mathbf{R} \longrightarrow \mathbf{R}$ given by $f(x) = \sin(x)$ is not bijective, hence does not have an inverse function. But if we let $A = [-\pi/2, \pi/2]$, the restriction $f|_A: A \longrightarrow [-1, 1]$ is bijective, so it has the inverse $\arcsin: [-1, 1] \longrightarrow A$.

Bounded functions

Definition 4.28. Let $E \subseteq \mathbf{R}$, $f: E \longrightarrow \mathbf{R}$.

We say that

- f is *bounded above* if the set $f(E)$ is bounded above in $\mathbf{R}$;
- f is *bounded below* if the set $f(E)$ is bounded below in $\mathbf{R}$;
- f is *bounded* if the set $f(E)$ is bounded in $\mathbf{R}$.

If $A \subseteq E$, we say that f is *bounded above on A* (resp. bounded below, resp. bounded) if the restriction $f|_A$ is bounded above, etc.

Example 4.29. Consider the function $f:\ (0,\infty) \longrightarrow \mathbf{R}$ given by $f(x) = \dfrac{1}{x}$.

Here $E = (0,\infty)$.

f is bounded below (by 0).

f is not bounded above, hence not bounded.

If $A = [1,\infty)$, then f is bounded above on A (by 1) and bounded below on A (by 0), hence it is bounded on A.

Lemma 4.30. *Let* $f : [a,b] \longrightarrow \mathbf{R}$ *be continuous. For every* $c \in (a,b)$ *there exists* $\delta > 0$ *such that* f *is bounded on the closed interval* $[c-\delta, c+\delta]$.

Proof. Take $\varepsilon = 1$. Since f is continuous at c, there exists $\delta' > 0$ such that if $x \in [a,b]$ and $|x - c| < \delta'$ then $|f(x) - f(c)| < 1$.

Take $\delta = \mathsf{min}\,\{\delta', c-a, b-c\}$, so that $0 < \delta \leqslant \delta'$ and $(c-\delta, c+\delta) \subseteq (a,b)$.

Take $C = \mathsf{max}\,\{|f(c)|+1, |f(c-\delta)|, |f(c+\delta)|\}$.

Let $x \in [c-\delta, c+\delta]$.

If $x = c-\delta$ then $|f(x)| = |f(c-\delta)| \leqslant C$. Similarly for $x = c+\delta$.

If $x \in (c-\delta, c+\delta)$, then $|f(x) - f(c)| < 1$ so

$$|f(x)| \leqslant |f(x) - f(c)| + |f(c)| < 1 + |f(c)| \leqslant C.$$

□

It is also the case that there exist $\delta_a, \delta_b > 0$ such that f is bounded on the closed intervals $[a, a+\delta_a]$ and $[b-\delta_b, b]$, see Exercise 4.11.

Theorem 4.31. *Let $f : [a, b] \longrightarrow \mathbf{R}$. If f is continuous, then it is bounded.*

Proof. Consider the set

$$B = \{x \in [a, b] : \ f \text{ is bounded on } [a, x]\}.$$

It is clear that B is bounded above by b.

By Exercise 4.11 there exists $\delta_a > 0$ such that f is bounded on $[a, a+\delta_a]$, so $a+\delta_a \in B$, hence B is nonempty. By the Completeness Axiom, $r = \sup B$ exists. Note that $a < a + \delta_a \leqslant r \leqslant b$.

It remains to show that $r \in B$ (that is, f is bounded on $[a, r]$) and $r = b$.

We know that $a < r \leqslant b$. By Lemma 4.30 and Exercise 4.11 there exists $\delta > 0$ such that f is bounded on $[r - \delta, r]$. Since $r = \sup B$, $r - \delta$ is not an upper bound of B, so there exists $c \in B$ such that $r - \delta < c \leqslant r$. Since $c \in B$, f is bounded on $[a, c]$, hence f is bounded on $[a, c] \cup [r - \delta, r] = [a, r]$. We conclude that $r \in B$.

To show that $r = b$, we proceed by contradiction: suppose $r < b$. Since $a < r < b$, by Lemma 4.30 there exists $\delta > 0$ such that f is bounded on $[r - \delta, r + \delta]$. Therefore f is bounded on $[a, r] \cup [r - \delta, r + \delta] = [a, r + \delta]$, so $r + \delta \in B$, contradicting the fact that $r = \sup B$. □

Theorem 4.32 (Extreme Value Theorem). *Let $f : [a,b] \longrightarrow \mathbf{R}$ be continuous. There exist $c, d \in [a,b]$ such that*

$$f(c) \leqslant f(x) \leqslant f(d) \qquad \textit{for all } x \in [a,b].$$

Proof. We prove that f attains its maximum value at some $d \in [a,b]$. (The case of the minimum is similar.)

Let $Y = f([a,b])$ be the image of f. Since f is continuous on $[a,b]$, it is bounded by Theorem 4.31, so Y is bounded. It is also non-empty (e.g. $f(a) \in Y$). By the Completeness Axiom, $r = \sup Y$ exists.

For each $n \geqslant 1$, $r - \frac{1}{n}$ is not an upper bound of Y, hence there exists $y_n \in Y$ such that

$$r - \frac{1}{n} < y_n \leqslant r.$$

Since Y is the image of f, this means that there exists $x_n \in [a,b]$ such that

$$r - \frac{1}{n} < f(x_n) \leqslant r.$$

(continued).

This gives us a sequence (x_n) in $[a, b]$. Since this sequence is bounded, the Bolzano–Weierstrass Theorem says that (x_n) has a convergent subsequence (x_{n_k}). Let d be the limit of (x_{n_k}) as $k \longrightarrow \infty$. We have $a \leqslant d \leqslant b$ by the Inequalities and Limits Theorem.

By the Sequential Criterion for Continuity, we have $f(x_{n_k}) \longrightarrow f(d)$ as $k \longrightarrow \infty$. But

$$r - \frac{1}{n_k} < f(x_{n_k}) \leqslant r \qquad \text{for all } k \in \mathbf{N},$$

so by the Sandwich Theorem for sequences we conclude that $f(d) = r = \sup Y$. Since $f(d) \in Y$, this implies that $f(x) \leqslant f(d)$ for all $x \in [a, b]$. □

Lemma 4.33. *Let $h : [a,b] \longrightarrow \mathbf{R}$ be continuous. If $h(a) < 0 < h(b)$, then there exists $c \in (a,b)$ such that $h(c) = 0$.*

Proof. Let $S = \{x \in [a,b] : \ h(t) \leqslant 0 \text{ for all } t \in [a,x]\}$. We have $a \in S$ since $h(a) < 0$, so S is non-empty. On the other hand, $h(b) > 0$ so $b \notin S$ and b is an upper bound for S in $\mathbf{R}$.

By the Completeness Axiom, the set S has a supremum c. We have $a \leqslant c \leqslant b$, in particular h is defined (and continuous) at c.

We use a proof by contradiction to show that $h(c) = 0$ (and therefore conclude).

Suppose $h(c) \neq 0$, then either $h(c) > 0$ or $h(c) < 0$.

- Suppose $h(c) > 0$. Set $\varepsilon = h(c) > 0$. Since h is continuous at c, there exists $\delta > 0$ such that for all $x \in \mathbf{R}$ we have

$$|x - c| < \delta \Rightarrow |h(x) - h(c)| < \varepsilon \Rightarrow 0 = h(c) - \varepsilon < h(x) < h(c) + \varepsilon.$$

  So $h(x) > 0$ for all $x \in (c - \delta, c + \delta)$. In particular, $h(c - \delta/2) > 0$, so $c - \delta/2 \notin S$ and is an upper bound of S, contradicting $c = \sup S$.

(continued).

- Suppose $h(c) < 0$. Set $\varepsilon = -h(c) > 0$. Since h is continuous at c, there exists $\delta > 0$ such that for all $x \in \mathbf{R}$ we have

$$|x - c| < \delta \Rightarrow |h(x) - h(c)| < \varepsilon \Rightarrow h(c) - \varepsilon < h(x) < h(c) + \varepsilon = 0.$$

  So $h(x) < 0$ for all $x \in (c - \delta, c + \delta)$. Since $c - \delta$ is not an upper bound of S, there exists $x \in S$ with $x > c - \delta$. We have $h(t) \leqslant 0$ for all $t \in [a, x]$ and $h(t) < 0$ for all $t \in (c - \delta, c + \delta)$, so $h(t) \leqslant 0$ for all $t \in [a, c + \delta)$. In particular, $c + \delta/2 \in S$, contradicting $c = \sup S$. □

Theorem 4.34 (Intermediate Value Theorem)**.** *Let* $f: [a,b] \longrightarrow \mathbf{R}$ *be continuous. If* $y \in \mathbf{R}$ *is strictly between* $f(a)$ *and* $f(b)$*, then there exists* $c \in (a,b)$ *such that* $f(c) = y$*.*

Proof. WLOG $f(a) < y < f(b)$ (otherwise interchange the roles of a and b).

Let $h: [a,b] \longrightarrow \mathbf{R}$ be given by $h(x) = f(x) - y$. Then h is continuous (by the Algebra of Continuity Theorem) and $h(a) < 0 < h(b)$. Hence by Lemma 4.33 there exists $c \in (a,b)$ such that $0 = h(c) = f(c) - y$, so $f(c) = y$. □

Example 4.35. Let $f : \mathbf{R} \longrightarrow \mathbf{R}$ be given by $f(x) = -2x^3 + 3x + 5$.

Does f have any real roots?

It is a continuous function with $f(0) = 5 > 0$ and $f(2) = -5 < 0$. Apply the Intermediate Value Theorem to the restriction of f to the closed interval $[0, 2]$ to get that there is at least one real root.

4.3 Sequences of functions

Recall that a sequence (of real numbers) is a function $a: \mathbf{N} \longrightarrow \mathbf{R}$. We think of it as (a_n), where for each $n \in \mathbf{N}$ there is a real number a_n.

Definition 4.36. Fixing a subset $E \subseteq \mathbf{R}$, consider the set

$$\mathcal{F} = \{\text{all functions } E \longrightarrow \mathbf{R}\}.$$

Then a *sequence of functions* is a function $f: \mathbf{N} \longrightarrow \mathcal{F}$. We think of it as (f_n), where for each $n \in \mathbf{N}$ there is a function $f_n: E \longrightarrow \mathbf{R}$.

Example 4.37. For all $n \geqslant 1$ let $f_n: \mathbf{R} \longrightarrow \mathbf{R}$ be given by the formula

$$f_n(x) = \frac{2x^2 + 2nx}{n}.$$

We have

$$f_3(x) = \frac{2x^2 + 6x}{3}, \qquad f_{10}(x) = \frac{2x^2 + 20x}{10}, \qquad \ldots$$

For every fixed $n \geqslant 1$, we get a function $f_n: \mathbf{R} \longrightarrow \mathbf{R}$.

(continued.)

If we fix $x \in \mathbf{R}$, say $x = 1$, we get a sequence of real numbers

$$\big(f_n(1)\big) = \big(f_1(1), f_2(1), f_3(1), \dots\big) = \left(\frac{2+2n}{n}\right).$$

Note that $\lim\limits_{n\to\infty} f_n(1) = 2$.

In general, for any fixed $x \in \mathbf{R}$, we have

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{2x^2 + 2nx}{n} = \lim_{n\to\infty}\left(\frac{2x^2}{n} + 2x\right) = 2x.$$

We give a name to the behaviour in the example.

Definition 4.38. Let $E \subseteq \mathbf{R}$ and let (f_n) be a sequence of functions $f_n : E \longrightarrow \mathbf{R}$. Let $f : E \longrightarrow \mathbf{R}$ be a function.

We say that (f_n) *converges pointwise* to f if: for every $x \in E$ we have $f_n(x) \longrightarrow f(x)$.

In more detail:

$$(\forall x \in E)(\forall \varepsilon > 0)(\exists M \in \mathbf{N}) n > M \Rightarrow |f_n(x) - f(x)| < \varepsilon.$$

Very important observation: the natural number M in the definition depends on both ε and x.

(In other words, if we take a different $x \in E$, we may have to change M.)

Example 4.39. Back to the sequence $f_n(x) = \dfrac{2x^2 + 2nx}{n}$.

For $x = 1$, $f_n(1) = \frac{2+2n}{n}$. Let $\varepsilon = 1/2$. We want $M \in \mathbf{N}$ such that if $n > M$ then

$$|f_n(1) - 2| = \left|\frac{2 + 2n}{n} - 2\right| = \frac{2}{n} < \varepsilon = \frac{1}{2}.$$

So we can take $M = 4$ (and this is the smallest M that works here).

For $x = 100$, we have $f_n(100) = \frac{20000+200n}{n}$. Let $\varepsilon = 1/2$. We want $M \in \mathbf{N}$ such that if $n > M$ then

$$|f_n(100) - 200| = \left|\frac{20000 + 200n}{n} - 200\right| = \frac{20000}{n} < \varepsilon = \frac{1}{2}.$$

So we can take $M = 40000$ (and this is the smallest M that works here; definitely $M = 4$ does not work).

The fact that in the case of pointwise convergence we may need different M's for different points x in the domain causes a lot of trouble, in that it makes it very difficult to say anything precise about the limit function f.

Therefore, a stronger notion of convergence of sequences of functions is generally preferred:

Definition 4.40. Let $E \subseteq \mathbf{R}$ and let (f_n) be a sequence of functions $f_n : E \longrightarrow \mathbf{R}$. Let $f : E \longrightarrow \mathbf{R}$ be a function.

We say that (f_n) *converges uniformly* to f if:

$$(\forall \varepsilon > 0)(\exists M \in \mathbf{N})(\forall x \in E) n > M \Rightarrow |f_n(x) - f(x)| < \varepsilon.$$

Very important observation: simply moving the quantifier $(\forall x \in E)$ from the start of the statement to right after $(\exists M \in \mathbf{N})$ has a profound consequence. Now M depends only on ε, not on the particular $x \in E$.

In other words, the same M works for all $x \in E$.

Example 4.41. For $n \geqslant 1$, let $f_n : \mathbf{R} \longrightarrow \mathbf{R}$ be given by $f_n(x) = \dfrac{1}{n(1+x^2)}$.

Clearly the pointwise limit is the constant function zero on $\mathbf{R}$.

To see that the convergence is uniform, note that $1 + x^2 \geqslant 1$ for all $x \in \mathbf{R}$, so

$$0 \leqslant \frac{1}{1+x^2} \leqslant 1 \quad \Rightarrow \quad 0 \leqslant f_n(x) \leqslant \frac{1}{n} \quad \text{for all } n \geqslant 1.$$

Now let $\varepsilon > 0$. Take $M = \lceil 1/\varepsilon \rceil$. If $n > M$ we have, for all $x \in \mathbf{R}$:

$$|f_n(x) - 0| \leqslant \frac{1}{n} < \frac{1}{N} \leqslant \varepsilon.$$

So (f_n) converges uniformly.

Theorem 4.42 (Uniform Limit Continuity Theorem)**.** *Let $E \subseteq \mathbf{R}$ and let (f_n) be a sequence of continuous functions $f_n : E \longrightarrow \mathbf{R}$. Let $f : E \longrightarrow \mathbf{R}$. If (f_n) converges uniformly to f, then f is continuous.*

Proof. Let $a \in E$. Let $\varepsilon > 0$. Take $\varepsilon' = \varepsilon/3$.

Since (f_n) converges uniformly to f, there exists $M \in \mathbf{N}$ such that if $n > M$, for all $x \in E$ we have

$$|f_n(x) - f(x)| < \varepsilon'.$$

In particular, for all $x \in E$ we have

$$|f_{M+1}(x) - f(x)| < \varepsilon' \quad \text{and} \quad |f_{M+1}(a) - f(a)| < \varepsilon'.$$

But f_{M+1} is continuous at a, so there exist $\delta > 0$ such that if $x \in E$ such that $|x - a| < \delta$, then

$$|f_{M+1}(x) - f_{M+1}(a)| < \varepsilon'.$$

Now let $x \in E$ such that $|x - a| < \delta$, then

$$\begin{aligned} |f(x) - f(a)| &= |f(x) + f_{M+1}(x) - f_{M+1}(x) + f_{M+1}(a) - f_{M+1}(a) - f(a)| \\ &\leqslant |f(x) - f_{M+1}(x)| + |f_{M+1}(x) - f_{M+1}(a)| + |f_{M+1}(a) - f(a)| \\ &< \varepsilon' + \varepsilon' + \varepsilon' = \varepsilon. \end{aligned}$$

We conclude that f is continuous at every $a \in E$. □

The Uniform Limit Continuity Theorem can be used to prove that a sequence does **not** converge uniformly.

Example 4.43. For $n \geqslant 1$, let $f_n : [0,1] \longrightarrow \mathbf{R}$ be given by $f_n(x) = x^n$.

What is the pointwise limit of the sequence (f_n)?

If $0 \leqslant x < 1$ then $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = 0$.

For $x = 1$ we have $\lim_{n\to\infty} f_n(1) = \lim_{n\to\infty} 1 = 1$.

So the limit function is $f : [0,1] \longrightarrow \mathbf{R}$ given by

$$f(x) = \begin{cases} 0 & \text{if } 0 \leqslant x < 1, \\ 1 & \text{if } x = 1. \end{cases}$$

Since the limit function f is not continuous, we conclude that the sequence (f_n) does not converge uniformly.

Beware: if the limit function f turns out to be continuous, that does not necessarily imply that the convergence is uniform. See Exercise 4.15 for a counterexample.

Another property that is preserved by uniform convergence but not by pointwise convergence is boundedness, see Exercises 4.16 and 4.17.