4 Functional limits and continuity

Chapter contents

4.1 Limits of functions

We shift our attention from limits of sequences (that is, functions $\mathbf{N} \longrightarrow \mathbf{R}$) as the variable n tends to ∞, to limits of functions $\mathbf{R} \longrightarrow \mathbf{R}$ as the variable x tends to some point $a \in \mathbf{R}$ (or, a little later, to ∞).

For instance, you may have seen that for the function $f: \mathbf{R} \longrightarrow \mathbf{R}$ defined by $f(x) = 2x$, we have that $f(x)$ tends to 6 as $x \to 3$:

x	$f(x) = 2x$
4	
3.5	
3.1	
3.01	
3.001	
etc.	

The definition of limit of a function should be flexible enough to cover functions that are much less well-behaved than $f(x) = 2x$. We will consider some of the issues that can arise.

Example 4.1. Let $f:\ \mathbf{R} \longrightarrow \mathbf{R}$ be given by

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbf{Q}, \\ -1 & \text{otherwise.} \end{cases}$$

x	$f(x)$	x	$f(x)$
1		$\sqrt{2} - 0.1$	
1.4		$\sqrt{2} - 0.01$	
1.41		$\sqrt{2} - 0.001$	
1.414		$\sqrt{2} - 0.0001$	
1.4142		$\sqrt{2} - 0.00001$	

On the other hand, jumps in the graph are **not necessarily** a dealbreaker:

Example 4.2. Let $f : \mathbf{R} \longrightarrow \mathbf{R}$ be given by

$$f(x) = \begin{cases} 1 & \text{if } x \neq 3, \\ -1 & \text{otherwise.} \end{cases}$$

How about for $g : \mathbf{R} \longrightarrow \mathbf{R}$ given by

$$g(x) = \begin{cases} 1 & \text{if } x < 3, \\ -1 & \text{otherwise?} \end{cases}$$

Many functions are not defined on all of $\mathbf{R}$

Let $E \subseteq \mathbf{R}$ be an arbitrary subset. At what points a would it make sense to talk about limits for functions $f: E \longrightarrow \mathbf{R}$?

We cannot just take any $a \in \mathbf{R}$:

On the other hand, restricting to $a \in E$ is not quite right either:

Limit points of sets

Definition 4.3. Let $E \subseteq \mathbf{R}$. An element $a \in \mathbf{R}$ is a *limit point of E* if for every $\delta > 0$ there exists $x \in E$ such that

$$0 < |x - a| < \delta.$$

Example 4.4. Consider the function $f: (0, \infty) \longrightarrow \mathbf{R}$ given by $f(x) = \log(x)$.

Example 4.5. What is the set of limit points of $E = \mathbf{N} \subseteq \mathbf{R}$?

Theorem 4.6. *Let $E \subseteq \mathbf{R}$ and $a \in \mathbf{R}$. Then a is a limit point of E if and only if there exists a sequence (x_n) such that $x_n \longrightarrow a$, and for every $n \in \mathbf{N}$ we have $x_n \in E$ and $x_n \neq a$.*

Limit of a function

Definition 4.7. Let $E \subseteq \mathbf{R}$, let a be a limit point of E, let $f : E \longrightarrow \mathbf{R}$ be a function, and let $L \in \mathbf{R}$.

We say that L is the *limit of f as $x \longrightarrow a$* and write

$$\lim_{x \to a} f(x) = L \qquad \text{or} \qquad f(x) \longrightarrow L \text{ as } x \longrightarrow a,$$

if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in E$ and $0 < |x - a| < \delta$, then $|f(x) - L| < \varepsilon$.

Example 4.8. For any $a, c \in \mathbf{R}$ we have

$$\lim_{x \to a} x = a \qquad \text{and} \qquad \lim_{x \to a} c = c.$$

Example 4.9.

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = 4.$$

Scrap work. Fix $\varepsilon > 0$ and "solve for" δ:

Proof.

Theorem 4.10 (Sequential Criterion for Function Limits)**.** *Let* $E \subseteq \mathbf{R}$*,* $a \in \mathbf{R}$ *a limit point of* E*,* $f: E \longrightarrow \mathbf{R}$*, and* $L \in \mathbf{R}$*. Then*

$$\lim_{x \to a} f(x) = L$$

if and only if:

$(*)$ *for every sequence* (a_n) *such that* $a_n \in E \setminus \{a\}$ *for all* $n \in \mathbf{N}$ *and* $a_n \longrightarrow a$*, we have that* $f(a_n) \longrightarrow L$*.*

Theorem 4.11 (Algebra of Limits). *Let $E \subseteq \mathbf{R}$, $a \in \mathbf{R}$ a limit point of E, and $f, g : E \longrightarrow \mathbf{R}$ such that*

$$\lim_{x \to a} f(x) = \alpha \quad \textit{and} \quad \lim_{x \to a} g(x) = \beta.$$

Then

(a) $\lim_{x \to a} f(x) + g(x) = \alpha + \beta$;

(b) $\lim_{x \to a} f(x)g(x) = \alpha\beta$;

(c) $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\alpha}{\beta}$ *if* $\beta \neq 0$.

Theorem 4.12 (Inequalities and Limits)**.** *Let $E \subseteq \mathbf{R}$, $a \in \mathbf{R}$ a limit point of E, and $f, g : E \longrightarrow \mathbf{R}$ such that*

$$\lim_{x \to a} f(x) = \alpha \quad \textit{and} \quad \lim_{x \to a} g(x) = \beta.$$

If $f(x) \leqslant g(x)$ for all $x \in E$, then $\alpha \leqslant \beta$.

Corollary 4.13. *With the same assumptions as in the Theorem:*

If $b \in \mathbf{R}$ is such that $b \leqslant f(x)$ for all $x \in E$, then $b \leqslant \alpha$.

If $c \in \mathbf{R}$ is such that $f(x) \leqslant c$ for all $x \in E$, then $\alpha \leqslant c$.

Example 4.14.

$$\lim_{x \to 1} \frac{x^2 - 1}{2x - 1} = 0.$$

Infinite limits and limits at infinity

Definition 4.15. Let $E \subseteq \mathbf{R}$, $a \in \mathbf{R}$ a limit point of E, and $f : E \longrightarrow \mathbf{R}$ a function. We write

$$\lim_{x \to a} f(x) = \infty$$

if for every $r \in \mathbf{R}$ there exists $\delta > 0$ such that if $x \in E$ and $0 < |x - a| < \delta$ then $f(x) > r$.

You should write down the analogous definition of $\lim_{x \to a} f(x) = -\infty$.

Example 4.16.

$$\lim_{x \to 0} \frac{1}{x^2} = \infty.$$

Definition 4.17. Let $E \subseteq \mathbf{R}$ be a set that is not bounded above, let $f : E \longrightarrow \mathbf{R}$ be a function, and let $L \in \mathbf{R}$. We write

$$\lim_{x \to \infty} f(x) = L$$

if for every $\varepsilon > 0$ there exists $M \in \mathbf{R}$ such that if $x \in E$ and $x > M$ then $|f(x) - L| < \varepsilon$.

You should write down the analogous definition of $\lim_{x \to -\infty} f(x) = L$.

Example 4.18.

$$\lim_{x\to\infty} \frac{1}{x^3} = 0.$$

4.2 Continuity

Definition 4.19. Let $E \subseteq \mathbf{R}$, $f: E \longrightarrow \mathbf{R}$, and $a \in E$. We say that f is *continuous at a* if: for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in E$ and $|x - a| < \delta$ then $|f(x) - f(a)| < \varepsilon$.

We say that f is *continuous on E* (or simply: *continuous*) if it is continuous at every point $a \in E$.

Example 4.20. Show that $f: \mathbf{R} \longrightarrow \mathbf{R}$ given by $f(x) = x$ is continuous.

You will prove that for any $c \in \mathbf{R}$, the constant function $f: \mathbf{R} \longrightarrow \mathbf{R}$ given by $f(x) = c$ for all $x \in \mathbf{R}$, is continuous, see

Example 4.21. Consider $f : \mathbf{R} \longrightarrow \mathbf{R}$ given by $f(x) = x^2$, with $a = 2$.

Let $E \subseteq \mathbf{R}$, $f: E \longrightarrow \mathbf{R}$, and $a \in E$. We have that f is **not** continuous at a if

Example 4.22. Consider the following function $f: \mathbf{R} \longrightarrow \mathbf{R}$ at $a = 0$:

$$f(x) = \begin{cases} 2x+1 & \text{if } x \neq 0, \\ 5 & \text{if } x = 0. \end{cases}$$

Theorem 4.23 (Limit Criterion for Continuity)**.** *Let $E \subseteq \mathbf{R}$, $f : E \longrightarrow \mathbf{R}$, and $a \in E$ be a limit point of E. The function f is continuous at a if and only if $\lim_{x \to a} f(x) = f(a)$.*

You will prove that if $a \in E$ is **not** a limit point of E, then any function $f : E \longrightarrow \mathbf{R}$ is continuous at a, see

Theorem 4.24 (Algebra of Continuity)**.** *Let* $E \subseteq \mathbf{R}$, $f, g: E \longrightarrow \mathbf{R}$, $a \in E$. *If* f *and* g *are both continuous at* a*, then*

(a) $f + g$ *is continuous at* a*;*

(b) fg *is continuous at* a*;*

(c) f/g *is continuous at* a *if* $g(a) \neq 0$.

Corollary 4.25. *All polynomial functions* $\mathbf{R} \longrightarrow \mathbf{R}$ *are continuous.*

Once these functions are properly defined, one can show that they are continuous (on their respective domains of definition): e^x, $\sin(x)$, $\cos(x)$, $\log(x)$, $x^{1/p}$ for $p \in \mathbf{N}_{\geqslant 1}$.

Theorem 4.26 (Sequential Criterion for Continuity)**.** *Let $E \subseteq \mathbf{R}$, $f:\ E \longrightarrow \mathbf{R}$, $a \in E$. The function f is continuous at a if and only if:*

$$(*) \quad \textit{for every sequence } (a_n) \textit{ in } E \textit{ such that } a_n \longrightarrow a \textit{ we have } f(a_n) \longrightarrow f(a).$$

Theorem 4.27 (Composition and Continuity)**.** *Let* $A, B \subseteq \mathbf{R}$, $f : A \longrightarrow \mathbf{R}$ *and* $g : B \longrightarrow \mathbf{R}$ *such that* $f(A) \subseteq B$. *Let* $a \in A$. *If* f *is continuous at* a *and* g *is continuous at* $f(a)$, *then* $g \circ f$ *is continuous at* a.

Let $E \subseteq \mathbf{R}$, $f: E \longrightarrow \mathbf{R}$.

For a subset $A \subseteq E$, let $f|_A: A \longrightarrow \mathbf{R}$ denote the *restriction of f to A*:

$$f|_A(x) = f(x) \qquad \text{for all } x \in A.$$

You will show that if $a \in A$ and f is continuous at a, then $f|_A$ is continuous at a, see

Bounded functions

Definition 4.28. Let $E \subseteq \mathbf{R}$, $f: \ E \longrightarrow \mathbf{R}$.

We say that

- f is *bounded above* if the set $f(E)$ is bounded above in $\mathbf{R}$;
- f is *bounded below* if the set $f(E)$ is bounded below in $\mathbf{R}$;
- f is *bounded* if the set $f(E)$ is bounded in $\mathbf{R}$.

If $A \subseteq E$, we say that f is *bounded above on A* (resp. bounded below, resp. bounded) if the restriction $f|_A$ is bounded above, etc.

Example 4.29. Consider the function $f : (0, \infty) \longrightarrow \mathbf{R}$ given by $f(x) = \dfrac{1}{x}$.

Lemma 4.30. *Let* $f : [a, b] \longrightarrow \mathbf{R}$ *be continuous. For every* $c \in (a, b)$ *there exists* $\delta > 0$ *such that* f *is bounded on the closed interval* $[c - \delta, c + \delta]$*.*

It is also the case that there exist $\delta_a, \delta_b > 0$ such that f is bounded on the closed intervals $[a, a+\delta_a]$ and $[b-\delta_b, b]$, see

Theorem 4.31. *Let $f : [a, b] \longrightarrow \mathbf{R}$. If f is continuous, then it is bounded.*

Theorem 4.32 (Extreme Value Theorem)**.** *Let* $f : [a, b] \longrightarrow \mathbf{R}$ *be continuous. There exist* $c, d \in [a, b]$ *such that*

$$f(c) \leqslant f(x) \leqslant f(d) \qquad \text{for all } x \in [a, b].$$

Lemma 4.33. *Let* $h : [a,b] \longrightarrow \mathbf{R}$ *be continuous. If* $h(a) < 0 < h(b)$*, then there exists* $c \in (a,b)$ *such that* $h(c) = 0$.

Theorem 4.34 (Intermediate Value Theorem)**.** *Let* $f : [a, b] \longrightarrow \mathbf{R}$ *be continuous. If* $y \in \mathbf{R}$ *is strictly between* $f(a)$ *and* $f(b)$*, then there exists* $c \in (a, b)$ *such that* $f(c) = y$*.*

Example 4.35. Let $f: \mathbf{R} \longrightarrow \mathbf{R}$ be given by $f(x) = -2x^3 + 3x + 5$.

Does f have any real roots?

4.3 Sequences of functions

Recall that a sequence (of real numbers) is a function $a : \mathbf{N} \longrightarrow \mathbf{R}$. We think of it as (a_n), where for each $n \in \mathbf{N}$ there is a real number a_n.

Definition 4.36. Fixing a subset $E \subseteq \mathbf{R}$, consider the set

$$\mathcal{F} = \{\text{all functions } E \longrightarrow \mathbf{R}\}.$$

Then a *sequence of functions* is a function $f : \mathbf{N} \longrightarrow \mathcal{F}$. We think of it as (f_n), where for each $n \in \mathbf{N}$ there is a function $f_n : E \longrightarrow \mathbf{R}$.

Example 4.37. For all $n \geqslant 1$ let $f_n : \mathbf{R} \longrightarrow \mathbf{R}$ be given by the formula

$$f_n(x) = \frac{2x^2 + 2nx}{n}.$$

We give a name to the behaviour in the example.

Definition 4.38. Let $E \subseteq \mathbf{R}$ and let (f_n) be a sequence of functions $f_n : E \longrightarrow \mathbf{R}$. Let $f : E \longrightarrow \mathbf{R}$ be a function.

We say that (f_n) *converges pointwise* to f if: for every $x \in E$ we have $f_n(x) \longrightarrow f(x)$.

In more detail:

$$(\forall x \in E)(\forall \varepsilon > 0)(\exists M \in \mathbf{N}) n > M \Rightarrow |f_n(x) - f(x)| < \varepsilon.$$

Very important observation: the natural number M in the definition depends on both ε and x.

(In other words, if we take a different $x \in E$, we may have to change M.)

Example 4.39. Back to the sequence $f_n(x) = \dfrac{2x^2 + 2nx}{n}$.

The fact that in the case of pointwise convergence we may need different M's for different points x in the domain causes a lot of trouble, in that it makes it very difficult to say anything precise about the limit function f.

Therefore, a stronger notion of convergence of sequences of functions is generally preferred:

Definition 4.40. Let $E \subseteq \mathbf{R}$ and let (f_n) be a sequence of functions $f_n : E \longrightarrow \mathbf{R}$. Let $f : E \longrightarrow \mathbf{R}$ be a function.

We say that (f_n) *converges uniformly* to f if:

$$(\forall \varepsilon > 0)(\exists M \in \mathbf{N})(\forall x \in E) n > M \Rightarrow |f_n(x) - f(x)| < \varepsilon.$$

Very important observation: simply moving the quantifier $(\forall x \in E)$ from the start of the statement to right after $(\exists M \in \mathbf{N})$ has a profound consequence. Now M depends only on ε, not on the particular $x \in E$.

In other words, the same M works for all $x \in E$.

Example 4.41. For $n \geqslant 1$, let $f_n : \mathbf{R} \longrightarrow \mathbf{R}$ be given by $f_n(x) = \dfrac{1}{n(1+x^2)}$.

Theorem 4.42 (Uniform Limit Continuity Theorem)**.** *Let $E \subseteq \mathbf{R}$ and let (f_n) be a sequence of continuous functions $f_n : E \longrightarrow \mathbf{R}$. Let $f : E \longrightarrow \mathbf{R}$. If (f_n) converges uniformly to f, then f is continuous.*

The Uniform Limit Continuity Theorem can be used to prove that a sequence does **not** converge uniformly.

Example 4.43. For $n \geqslant 1$, let $f_n : [0,1] \longrightarrow \mathbf{R}$ be given by $f_n(x) = x^n$.

Beware: if the limit function f turns out to be continuous, that does not necessarily imply that the convergence is uniform.