5 Differentiation and integration

Derivatives

Exercise 5.1. Prove the linearity of the derivative, Theorem 5.5:

Let $I \subseteq \mathbf{R}$ be an interval, $c \in I$, $f, g : I \longrightarrow \mathbf{R}$ two functions that are differentiable at c. Then

(a) $f + g : I \longrightarrow \mathbf{R}$ is differentiable at c and $(f+g)'(c) = f'(c) + g'(c)$;

(b) for any $\lambda \in \mathbf{R}$, $\lambda f : I \longrightarrow \mathbf{R}$ is differentiable at c and $(\lambda f)'(c) = \lambda f'(c)$.

Exercise 5.2. Consider the following statement:

"Let $f : [a, b] \longrightarrow \mathbf{R}$ be a continuous function. The maximum of f occurs either at an endpoint, at a point where f is not differentiable, or at a point $d \in (a, b)$ where $f'(d) = 0$."

(a) Sketch three example functions, each of which illustrates one aspect of the statement.

(b) Explain how the statement follows from Theorem 5.9.

Exercise 5.3.

(a) Draw a picture of a function to illustrate what the Mean Value Theorem asserts.

(b) The Mean Value Theorem requires the function to be continuous on $[a, b]$. Draw a picture of a function that fails this condition so that the conclusion of the Mean Value Theorem does not hold True. Draw a picture of a function that fails this condition but the conclusion of the Mean Value Theorem holds True.

(c) The Mean Value Theorem requires the function to be differentiable on (a, b). Draw a picture of a function that fails this condition so that the conclusion of the Mean Value Theorem does not hold True. Draw a picture of a function that fails this condition but the conclusion of the Mean Value Theorem holds True.

Exercise 5.4. Where possible, apply the Mean Value Theorem on the following functions, using the interval $[-1, 4]$. If the hypotheses of the Mean Value Theorem hold true, find the value of c that the theorem says must exist.

(a) $f(x) = x^2$

(b) $g(x) = (x+1)^a$ where $a \geqslant 1$

(c) $h(x) = |x - 1|$

(d) $k(x) = \log(2x + 4)$.

Exercise 5.5. Let $f : [a, b] \longrightarrow \mathbf{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Prove that if for all $x \in (a, b)$ we have $f'(x) = 0$, then f is constant on $[a, b]$.

[**Hint**: Use the Mean Value Theorem.]

Exercise 5.6 (Carathéodory's Theorem)**.** Let $I \subseteq \mathbf{R}$ be an interval, $f: I \longrightarrow \mathbf{R}$ be a function, and $c \in I$. Then f is differentiable at c if and only if there exists a function $\varphi: I \longrightarrow \mathbf{R}$ that is continuous at c and satisfies

$$f(x) - f(c) = \varphi(x)\,(x - c) \qquad \text{for all } x \in I.$$

In the case that f is differentiable at c, $f'(c) = \varphi(c)$.

Exercise 5.7 (Chain Rule)**.** Let $I, J \subseteq \mathbf{R}$ be intervals and let $f: I \longrightarrow \mathbf{R}$ and $g: J \longrightarrow \mathbf{R}$ be functions such that $f(I) \subseteq J$. Let $c \in I$. If f is differentiable at c and g is differentiable at $f(c)$, then $g \circ f$ is differentiable at c and $(g \circ f)'(c) = g'(f(c))\, f'(c)$.

Here $g \circ f: I \longrightarrow \mathbf{R}$ is the composition defined by $(g \circ f)(x) = g(f(x))$ for all $x \in I$. [**Hint**: Use Carathéodory's Theorem, Exercise 5.6.]

Exercise 5.8 (Inverse Function Theorem)**.** Let A, B be intervals and let $f: A \longrightarrow B$ be an invertible function with inverse $f^{-1}: B \longrightarrow A$. Let $b = f(a)$ with $a \in A$, $b \in B$.

Suppose f is differentiable at a with $f'(a) \neq 0$, and f^{-1} is continuous at b. Prove that f^{-1} is differentiable at b and

$$\left(f^{-1}\right)'(b) = \frac{1}{f'(a)}.$$

[**Hint**: Use Carathéodory's Theorem, see Exercise 5.6.]

Exercise 5.9. Let $g = \log: (0, \infty) \longrightarrow \mathbf{R}$ be the inverse of the exponential function $f = \exp: \mathbf{R} \longrightarrow (0, \infty)$. Prove that g is differentiable at every $b \in (0, \infty)$ with derivative

$$g'(b) = \frac{1}{b}.$$

(You may assume that f is differentiable on $\mathbf{R}$ with derivative $f' = f$.)

Integrals

Exercise 5.10. Let $f: [-2, 4] \longrightarrow \mathbf{R}$ be defined by $f(x) = -x + 3$. Consider the partition $P = \{-2, -1, 1, 4\}$.

(a) Find the upper sum $U(f, P)$.

(b) Find the lower sum $L(f, P)$.

(c) Find $\displaystyle\int_{-2}^{4} f(x)\, dx$ using the Fundamental Theorem of Calculus.

(d) Do your answers satisfy the correct inequality?

(e) Take a refinement of P and recalculate the lower and upper sums. Do all your numbers still line up?

Exercise 5.11. Let $f : [a,b] \longrightarrow \mathbf{R}$ be a bounded function. For each $n \in \mathbf{N}$, let P_n be a partition of $[a,b]$. If

$$\lim_{n\to\infty} \big(U(f,P_n) - L(f,P_n)\big) = 0,$$

then f is integrable on $[a,b]$ and

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} L(f,P_n) = \lim_{n\to\infty} U(f,P_n).$$

[**Hint**: Use Theorem 5.27.]

Exercise 5.12. Consider the function $f : [0,1] \longrightarrow \mathbf{R}$ given by $f(x) = x^2$.

(a) Fix $n \in \mathbf{N}$ with $n \geqslant 1$ and let $x_k = k/n$ for all k with $0 \leqslant k \leqslant n$. Let $P_n = \{x_0, x_1, \ldots, x_n\}$. Compute $L(f,P_n)$ and $U(f,P_n)$.

(b) Use the result of part (a) and Exercise 5.11 to prove that f is integrable and find the value of the integral.

With a bit more algebraic fortitude, one can use this argument for $f : [a,b] \longrightarrow \mathbf{R}$ given by $f(x) = x^2$.

Exercise 5.13. Let $E \subseteq \mathbf{R}$ and let $f, g : E \longrightarrow \mathbf{R}$ be bounded functions.

(a) Prove that

$$\begin{aligned} \sup\{f(x)+g(x) : x \in E\} &\leqslant \sup\{f(x) : x \in E\} + \sup\{g(x) : x \in E\}, \\ \inf\{f(x)+g(x) : x \in E\} &\geqslant \inf\{f(x) : x \in E\} + \inf\{g(x) : x \in E\}. \end{aligned}$$

(b) Let $\lambda \in \mathbf{R}$. Prove that

$$\begin{aligned} \lambda \geqslant 0 \quad &\Rightarrow \quad \sup\{\lambda f(x) : x \in E\} = \lambda \sup\{f(x) : x \in E\} \\ &\text{and} \quad \inf\{\lambda f(x) : x \in E\} = \lambda \inf\{f(x) : x \in E\}, \\ \lambda < 0 \quad &\Rightarrow \quad \sup\{\lambda f(x) : x \in E\} = \lambda \inf\{f(x) : x \in E\} \\ &\text{and} \quad \inf\{\lambda f(x) : x \in E\} = \lambda \sup\{f(x) : x \in E\}. \end{aligned}$$

(c) Suppose $f(x) \leqslant g(x)$ for all $x \in E$. Prove that

$$\begin{aligned} \sup\{f(x) : x \in E\} &\leqslant \sup\{g(x) : x \in E\}, \\ \inf\{f(x) : x \in E\} &\leqslant \inf\{g(x) : x \in E\}. \end{aligned}$$

(Feel free to only do one of each pair of statements above, as the proof for the other member of the pair is almost identical.)

Exercise 5.14. Let $f, g : [a, b] \longrightarrow \mathbf{R}$ be bounded functions, let $\lambda \in \mathbf{R}$, and let P be a partition of $[a, b]$. Use Exercise 5.13 to prove the following:

(a) $U(f+g, P) \leqslant U(f, P) + U(g, P)$ and $L(f+g, P) \geqslant L(f, P) + L(g, P)$; what can you say about $U(f+g)$ and $L(f+g)$?

(b) if $\lambda \geqslant 0$ then $U(\lambda f, P) = \lambda U(f, P)$ and $L(\lambda f, P) = \lambda L(f, P)$; if $\lambda < 0$ then $U(\lambda f, P) = \lambda L(f, P)$ and $L(\lambda f, P) = \lambda U(f, P)$; what can you say about $U(\lambda f)$ and $L(\lambda f)$?

(c) $U(f, P) \leqslant U(g, P)$ and $L(f, P) \leqslant L(g, P)$; what can you say about the relation between $U(f)$ and $U(g)$, and between $L(f)$ and $L(g)$?

Exercise 5.15. Prove the properties of the Riemann integral stated in Theorem 5.29 parts (b), (c), and (e).

[**Hint**: You can avoid working hard by using Exercise 5.14.]

Exercise 5.16. Assume that the following statement is `True` (which is indeed the case, but we will not prove it):

"Let $f : [a, b] \longrightarrow [c, d]$ be integrable and $\varphi : [c, d] \longrightarrow \mathbf{R}$ be continuous. Then the composition $\varphi \circ f : [a, b] \longrightarrow \mathbf{R}$ is integrable."

Use this to prove the following:

(a) (Part (d) of Theorem 5.29.) If $f : [a, b] \longrightarrow \mathbf{R}$ is integrable, then $|f| : [a, b] \longrightarrow \mathbf{R}$ is integrable and

$$\left| \int_a^b f(x)\, dx \right| \leqslant \int_a^b |f(x)|\, dx.$$

(b) If $f : [a, b] \longrightarrow \mathbf{R}$ is integrable, then $f^2 : [a, b] \longrightarrow \mathbf{R}$ is integrable.

(c) If $f, g : [a, b] \longrightarrow \mathbf{R}$ are integrable, then $fg : [a, b] \longrightarrow \mathbf{R}$ is integrable.

(d) If $f : [a, b] \longrightarrow \mathbf{R}$ is integrable, then $f^n : [a, b] \longrightarrow \mathbf{R}$ is integrable for all $n \in \mathbf{N}$.

Exercise 5.17. Determine, without proof, which of the functions defined below are integrable on $[-1, 1]$.

(a) $f(x) = x^7 - 5x^3$

(b) $g(x) = \sin(e^x - 12)$

(c) $h(x) = \begin{cases} x^2 & x \in \mathbf{Q} \\ -x^2 & x \notin \mathbf{Q} \end{cases}$

(d) $p(x) = \begin{cases} 0 & x = 0 \\ \frac{1}{x} & x \neq 0 \end{cases}$

(e) $q(x) = \begin{cases} x^2 + 2 & x < \frac{1}{3} \\ 9 - \log x & x \geqslant \dfrac{1}{3} \end{cases}$

(f) $r(x) = \begin{cases} 17 & x = -\frac{1}{2} \\ \dfrac{1}{3x^2 + 5} & x \neq -\frac{1}{2} \end{cases}.$

Exercise 5.18. Use Theorem 5.27 to prove that the function $f : [-1, 1] \longrightarrow \mathbf{R}$ given by

$$f(x) = \begin{cases} 0 & x \leqslant 0 \\ 1 & x > 0 \end{cases}$$

is integrable.

[**Hint**: Given $\varepsilon > 0$, pick a partition $P = \{-1, 0, a, 1\}$ with a chosen in such a way that $U(f, P) - L(f, P) < \varepsilon$.]

Exercise 5.19. Prove that the function $f : [-1, 1] \longrightarrow \mathbf{R}$ given by

$$f(x) = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases}$$

is integrable.

[**Hint**: Use the same strategy as in Exercise 5.18.]

Exercise 5.20. Look up the definitions of the two types of improper integrals in Tutorial 11A.

Which of the following integrals are proper Riemann integrals, and which are improper?

(a) $\displaystyle\int_0^\infty \frac{1}{1 + x^2}\, dx$

(b) $\displaystyle\int_0^{\pi/2} \tan x\, dx$

(c) $\displaystyle\int_1^2 \frac{1}{3x - 1}\, dx$

(d) $\displaystyle\int_0^2 \frac{1}{3x - 1}\, dx$

(e) $\displaystyle\int_{-\infty}^\infty e^{-x^4}\, dx$

(f) $\displaystyle\int_{-1}^1 \frac{\sin x}{x^2}\, dx$

(g) $\displaystyle\int_0^1 t^{1/2} e^{e^t}\, dt$

(h) $\displaystyle\int_0^1 \frac{dt}{t - \frac{1}{2}}$

(i) $\displaystyle\int_0^{100} e^{\lfloor t \rfloor} dt$ where $\lfloor t \rfloor$ is the *floor function*.

Exercise 5.21. Look up the definitions of the two types of improper integrals in Tutorial 11A.

Evaluate the following improper integrals, or show they are divergent. Feel welcome to use tools for limits you are familiar with from previous studies.

(a) $\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\, dx$

(b) $\displaystyle\int_{-1}^1 \frac{1}{x^{2/3}}\, dx$

(c) $\displaystyle\int_{-\infty}^2 \frac{1}{3 - x}\, dx$

(d) $\displaystyle\int_1^\infty \frac{\log x}{x}\, dx$

(e) $\displaystyle\int_1^\infty \frac{\log x}{x^2}\, dx$

(f) $\displaystyle\int_0^1 \frac{1}{\sqrt{1 - x^2}}\, dx$

(g) $\displaystyle\int_0^5 \frac{1}{\sqrt{5 - x}}\, dx$

(h) $\displaystyle\int_0^\infty \frac{1}{(x+1)(x+2)}\, dx.$

Answers

Solution 5.1.

(a) Suppose that f and g are differentiable at c. That is, the limits

$$\lim_{h\to 0}\frac{f(c+h)-f(c)}{h} \text{ and } \lim_{h\to 0}\frac{g(c+h)-g(c)}{h}$$

exist and f and g are continuous at c. Then using limit laws, we see that

$$\begin{aligned}\lim_{h\to 0}\frac{(f+g)(c+h)-(f+g)(c)}{h} &= \lim_{h\to 0}\frac{f(c+h)+g(c+h)-f(c)-g(c)}{h}\\ &= \lim_{h\to 0}\frac{f(c+h)-f(c)}{h}+\frac{g(c+h)-g(c)}{h}\\ &= \lim_{h\to 0}\frac{f(c+h)-f(c)}{h}+\lim_{h\to 0}\frac{g(c+h)-g(c)}{h}\\ &= f'(c)+g'(c).\end{aligned}$$

So the limit exists and $f+g$ is differentiable at c. Further, $(f+g)'(c)=f'(c)+g'(c)$.

(b) Similarly f being differentiable at c implies the limit

$$\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$$

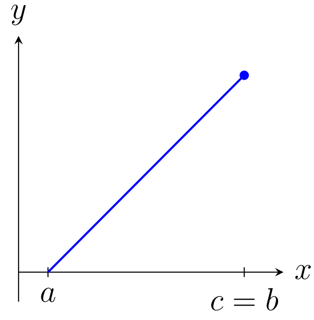
exists and f is continuous at c. Using limit laws, we see that

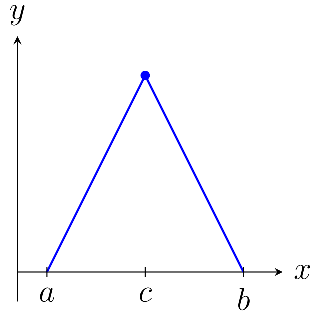
$$\begin{aligned}\lim_{h\to 0}\frac{(\lambda f)(c+h)-(\lambda f)(c)}{h} &= \lim_{h\to 0}\frac{\lambda f(c+h)-\lambda f(c)}{h} = \lim_{h\to 0}\lambda\frac{f(c+h)-f(c)}{h}\\ &= \lambda\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}\\ &= \lambda f'(c).\end{aligned}$$

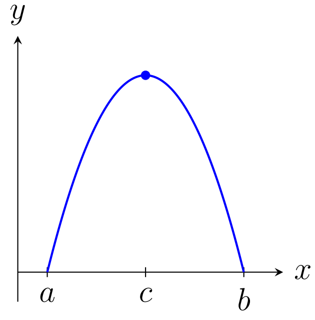
Hence λf is differentiable at c and $(\lambda f)'(c)=\lambda f'(c)$.

Solution 5.2.

(a) Here are some simple examples for each of the three behaviours:







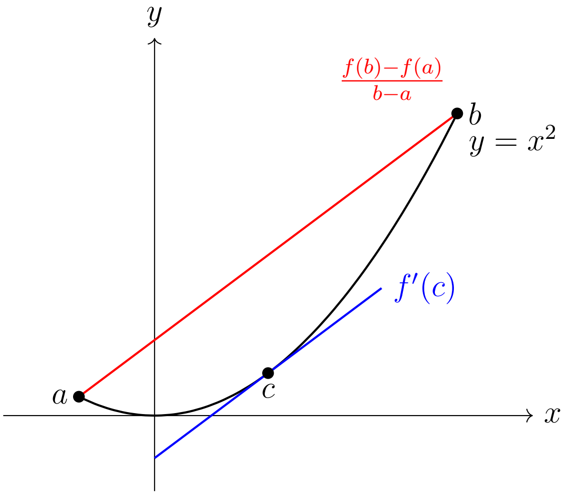
(b) We know that f attains its maximum at some point $d \in [a, b]$, by the Extreme Value Theorem. Suppose that d is not a boundary point (so $d \neq a, b$), and that f is differentiable at d. Letting $\delta = \min\{|a-d|, |b-d|\}$, we see that f has a local maximum at d, so by Theorem 5.9 we conclude that $f'(d) = 0$.

Solution 5.3.

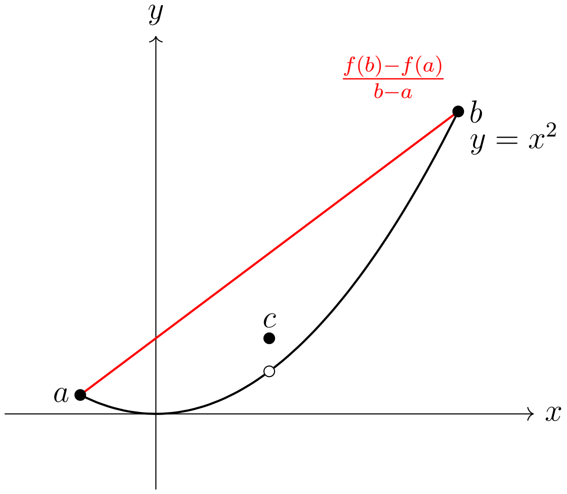
(a) In the following figure, the slope of the red line is

$$\frac{f(b)-f(a)}{b-a}.$$

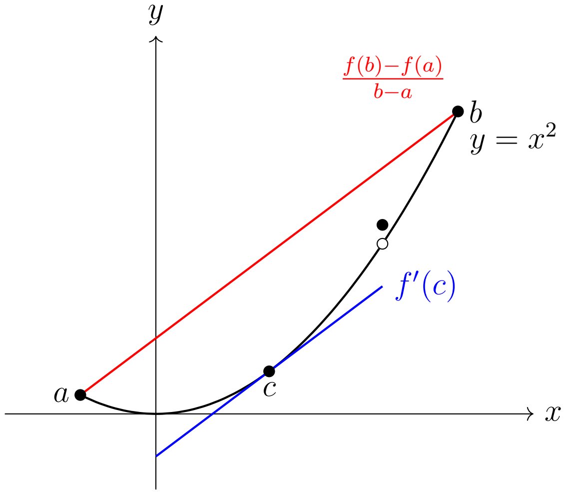
The Mean Value Theorem guarantees us a $c \in (a, b)$ such that the slope of f at c, indicated by the blue line, is the same as $\frac{f(b)-f(a)}{b-a}$. This is seen by the fact that the red line and blue line are parallel.



(b) To see when the implication is False, we can just remove this point c. Note that this works here because we know in this example, c is unique. To see when the implication remains True, remove some other point.

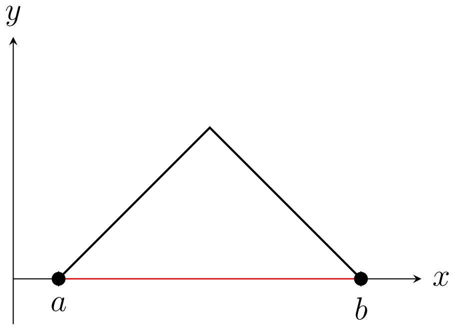


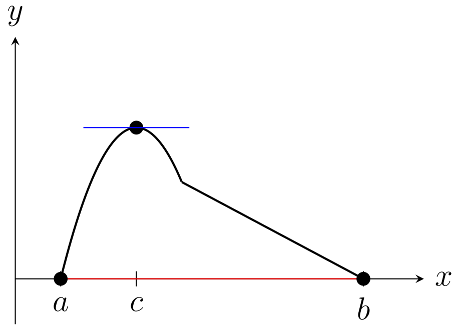
The implication fails.



The implication still holds.

(c) The above two examples can also be used here. Alternatively, here are some examples that are continuous:





Solution 5.4.

(a) The hypotheses hold, so the MVT says that there exists $c \in (-1,4)$ such that $f'(c) = 3$. Solving $2c = 3$ we get $c = 3/2$.

(b) The hypotheses hold, so the MVT says that there exists $c \in (-1,4)$ such that $g'(c) = 5^{a-1}$. Solving $a(c+1)^{a-1} = 5^{a-1}$ gives $c = \dfrac{5}{a^{\frac{1}{a-1}}} - 1$.

(c) The MVT does not apply to the function h because it is not differentiable at $x = 1$.

(d) The hypotheses hold, so the MVT says that there exists $c \in (-1,4)$ such that $k'(c) = \dfrac{\log(6)}{5}$. Solving $\dfrac{1}{c+2} = \dfrac{\log(6)}{5}$ gives $c = \dfrac{5}{\log(6)} - 2$.

Solution 5.5. To show f is constant on $[a, b]$, we can show that $f(x) = f(a)$ for all $x \in (a, b]$.

Let $x \in (a, b]$. By the Mean Value Theorem, we have that there exists $c \in (a, x)$ such that

$$\frac{f(a) - f(x)}{a - x} = f'(c).$$

Since $f'(x) = 0$ for all $x \in (a, b)$, we have that $f'(c) = 0$. Therefore

$$f(a) - f(x) = 0.$$

Solution 5.6.

"$\Longrightarrow$": Let f be differentiable at c. Then the limit

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

exists and is equal to $f'(c)$. Define

$$\varphi(x) = \begin{cases} \dfrac{f(x) - f(c)}{x - c} & \text{if } x \neq c, \\ f'(c) & \text{if } x = c. \end{cases}$$

Then one sees that $\lim_{x \to c} \varphi(x) = f'(c) = \varphi(c)$, so φ is continuous at c. Moreover, $\varphi(x)$ satisfies $f(x) - f(c) = \varphi(x)(x - c)$ for all $x \in I$ by definition.

" $\Longleftarrow$ ": Suppose f is a function and there exists a function $\varphi : I \longrightarrow \mathbf{R}$ that is continuous at c and satisfies $f(x) - f(c) = \varphi(x)(x - c)$ for all $x \in I$. By definition, $\varphi(x)$ being continuous at c implies that

$$\lim_{x \to c} \varphi(x) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

exists and is equal to $\varphi(c)$. So again by definition of differentiability, existence of this limit implies that f is differentiable at c. Moreover, the limit $\varphi(c) = f'(c)$.

Solution 5.7. We use Carathéodory's Theorem. Suppose f is differentiable at c. Then there exists a function $\varphi : I \longrightarrow \mathbf{R}$ that is continuous at c and satisfies

$$f(x) - f(c) = \varphi(x)(x - c) \quad \text{for all } x \in I.$$

Suppose g is differentiable at $f(c)$. Then there exists a function $\psi : J \longrightarrow \mathbf{R}$ that is continuous at $f(c)$ and satisfies

$$g(y) - g\big(f(c)\big) = \psi(y)\big(y - f(c)\big) \quad \text{for all } y \in J.$$

Moreover, $f'(c) = \varphi(c)$ and $\psi(c) = g'(c)$.

As $f(I) \subseteq J$, one has for all $x \in I$:

$$g\big(f(x)\big) - g\big(f(c)\big) = \psi\big(f(x)\big)\big(f(x) - f(c)\big) = \psi\big(f(x)\big)\varphi(x)(x - c).$$

Since f is differentiable at c, it is continuous at c. Along with continuity of φ and ψ at c, one sees that $\psi(f(x))\varphi(x)$ is continuous at c. By Carathéodory's Theorem again, we have that $g \circ f$ is differentiable at c, and

$$(g \circ f)'(c) = \psi\big(f(c)\big)\varphi(c) = g'\big(f(c)\big)f'(c)$$

as required.

Solution 5.8. Denote $f(a) = b$. We prove this using Carathéodory's theorem. As f is differentiable at a, there exists $\varphi : \ A \longrightarrow \mathbf{R}$ such that φ is continuous at a with $\varphi(a) = f'(a)$, and satisfies $f(x) - f(a) = \varphi(x)(x - a)$ for all $a \in A$. Since $f^{-1}(y) \in A$, we have that for all $y \in B$:

$$f\big(f^{-1}(y)\big) - b = y - b = \varphi\big(f^{-1}(y)\big)\big(f^{-1}(y) - f^{-1}(b)\big).$$

Define $\psi : \ B \longrightarrow \mathbf{R}$ by

$$\psi(y) = \frac{1}{\varphi\big(f^{-1}(y)\big)}.$$

Then by continuity laws, ψ is continuous at b and satisfies

$$\psi(y)(y - b) = f^{-1}(y) - f^{-1}(b).$$

Hence $f^{-1}(y)$ is differentiable at b and its derivative is

$$\big(f^{-1}\big)'(b) = \psi(b) = \frac{1}{\varphi\big(f^{-1}(b)\big)} = \frac{1}{\varphi(a)} = \frac{1}{f'(a)}.$$

Solution 5.9. As f is invertible and it has nowhere zero derivative, by the inverse function theorem, $\log(x)$ is differentiable on all of $(0, \infty)$. Using that $f(\log(x)) = x$, we apply the chain rule.

$$\begin{aligned}
\frac{d}{dx} f\big(\log(x)\big) &= f'\big(\log(x)\big) \cdot \frac{d}{dx} \log(x) \\
&= f\big(\log(x)\big) \cdot \frac{d}{dx} \log(x) \\
&= x \frac{d}{dx} \log(x).
\end{aligned}$$

On the other hand,

$$\frac{d}{dx}\,f\big(\log(x)\big) = \frac{d}{dx}\,x = 1.$$

So together,

$$1 = x\,\frac{d}{dx}\,\log(x) \Rightarrow \frac{d}{dx}\,\log(x) = \frac{1}{x}.$$

Solution 5.10.

(a)

$$\begin{aligned} \mathsf{sup}\,\{f(x):\ x\in[-2,-1]\} &= 5\\ \mathsf{sup}\,\{f(x):\ x\in[-1,1]\} &= 4\\ \mathsf{sup}\,\{f(x):\ x\in[1,4]\} &= 2, \end{aligned}$$

therefore

$$U(f,P) = 5(-1+2) + 4(1+1) + 2(4-1) = 19.$$

(b) $L(f,P) = 5$.

(c) $\displaystyle\int_{-2}^{4} f(x)\,dx = 12$.

(d) $5 \leqslant 12 \leqslant 19$.

(e) This will depend on the refinement you choose.

Solution 5.11. Let $\varepsilon > 0$. By the hypothesis, there exists $M \in \mathbf{N}$ such that for all $n \in \mathbf{N}$, if $n > M$ then

$$U(f,P_n) - L(f,P_n) < \varepsilon.$$

By Theorem 5.27 we conclude that f is integrable.

For each $n \in \mathbf{N}$ we have

$$U(f,P_n) \geqslant U(f) = L(f) \geqslant L(f,P_n),$$

so that

$$0 \leqslant U(f,P_n) - L(f) \leqslant U(f,P_n) - L(f,P_n).$$

By the Sandwich Theorem we conclude that $U(f,P_n) - L(f) \longrightarrow 0$, so by the Algebra of Limits we have $U(f,P_n) \longrightarrow L(f)$.

Solution 5.12.

(a) Fix k with $1 \leqslant k \leqslant n$, so $[x_{k-1}, x_k] = [(k-1)/n, k/n]$. Since f is increasing on $[0,1]$ we have $m_k = (k-1)^2/n^2$ and $M_k = k^2/n^2$. Therefore

$$U(f, P_n) = \sum_{k=1}^{n} M_k(x_k - x_{k-1}) = \sum_{k=1}^{n} \frac{k^2}{n^3} = \frac{1}{n^3} \sum_{k=1}^{n} k^2 = \frac{(n+1)(2n+1)}{6n^2}.$$

Similarly

$$L(f, P_n) = \sum_{k=1}^{n} m_k(x_k - x_{k-1}) = \sum_{k=1}^{n} \frac{(k-1)^2}{n^3} = \frac{1}{n^3} \sum_{k=1}^{n} (k-1)^2 = \frac{(n-1)(2n-1)}{6n^2}.$$

(b) After a little manipulation we have that for all $n \geqslant 1$:

$$U(f, P_n) - L(f, P_n) = \frac{1}{n}.$$

Since this converges to 0 as $n \longrightarrow \infty$, we conclude from Exercise 5.11 that f is integrable and

$$\int_0^1 f(x)\,dx = \lim_{n\to\infty} U(f, P_n) = \lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n^2} = \frac{1}{3}.$$

Solution 5.13.

(a) Let $I_f = \{f(x) : \ x \in E\}$ and $I_g = \{g(x) : \ x \in E\}$ be the respective images of f and g. In Assignment Question 2.3 we have seen that

$$\sup (I_f + I_g) = \sup I_f + \sup I_g.$$

But

$$\{f(x) + g(x) : \ x \in E\} \subseteq I_f + I_g,$$

so

$$\sup \{f(x) + g(x) : \ x \in E\} \leqslant \sup (I_f + I_g) = \sup I_f + \sup I_g.$$

For the infimum, the same argument as in Assignment Question 2.3 gives

$$\inf (I_f + I_g) = \inf I_f + \inf I_g,$$

and the rest follows.

(b) Direct consequence of Exercise 2.17.

(c) Let $M_g = \sup \{g(x) : \ x \in E\}$. Then for all $x \in E$ we have $f(x) \leqslant g(x) \leqslant M_g$, in particular M_g is an upper bound for the set $\{f(x) : \ x \in E\}$. Therefore $\sup \{f(x) : \ x \in E\} \leqslant M_g$.

Solution 5.14. Write $P = \{x_0, x_1, \dots, x_n\}$.

(a) For $1 \leqslant k \leqslant n$, let $M_k(f) = \sup\{f(x) : \ x \in [x_{k-1}, x_k]\}$ and similarly for $M_k(g)$ and $M_k(f+g)$. By Exercise 5.13 we have

$$M_k(f+g) \leqslant M_k(f) + M_k(g),$$

so that

$$M_k(f+g)(x_k - x_{k-1}) \leqslant M_k(f)(x_k - x_{k-1}) + M_k(g)(x_k - x_{k-1}),$$

and summing over k gives

$$U(f+g, P) \leqslant U(f, P) + U(g, P).$$

We show that the inequality also holds for the upper Riemann integrals. For this, fix $\varepsilon > 0$. Since $U(f) = \inf\{U(f, P) : \ P\}$, there exists a partition P_f such that

$$U(f, P_f) < U(f) + \frac{\varepsilon}{2}.$$

Similarly, there exists a partition P_g such that

$$U(g, P_g) < U(g) + \frac{\varepsilon}{2}.$$

Let $P = P_f \cup P_g$ be the common refinement of P_f and P_g, then

$$U(f+g) \leqslant U(f+g, P) \leqslant U(f, P) + U(g, P) \leqslant U(f, P_f) + U(g, P_g) < U(f) + U(g) + \varepsilon.$$

Since this holds for all $\varepsilon > 0$, we conclude that

$$U(f+g) \leqslant U(f) + U(g).$$

The argument for the lower Riemann sums is similar and gives the opposite inequalities.

(b) Fix k with $1 \leqslant k \leqslant n$.

Suppose first that $\lambda \geqslant 0$. By Exercise 5.13 we have

$$M_k(\lambda f) = \lambda M_k(f),$$

so that after multiplying by $(x_k - x_{k-1})$ and summing over k we get

$$U(\lambda f, P) = \lambda U(f, P).$$

Taking the infimum over all partitions P we get

$$U(\lambda f) = \inf\{U(\lambda f, P) : \ P\} = \inf\{\lambda U(f, P) : \ P\} = \lambda \inf\{U(f, P) : \ P\} = \lambda U(f).$$

If $\lambda < 0$, Exercise 5.13 gives

$$M_k(\lambda f) = \lambda m_k(f),$$

which leads us to

$$U(\lambda f, P) = \lambda L(f, P).$$

Taking the infimum over all partitions P we get

$$U(\lambda f) = \inf\{U(\lambda f, P) : \ P\} = \inf\{\lambda L(f, P) : \ P\} = \lambda \sup\{L(f, P) : \ P\} = \lambda L(f).$$

The argument for $L(\lambda f)$ is similar.

(c) Fix k with $1 \leqslant k \leqslant n$.

By Exercise 5.13 we have

$$M_k(f) \leqslant M_k(g),$$

so that after multiplying by $(x_k - x_{k-1})$ and summing over k we get

$$U(f, P) \leqslant U(g, P).$$

Taking the infimum over all partitions P we get

$$U(f) \leqslant U(g).$$

The argument giving $L(f) \leqslant L(g)$ is similar.

Solution 5.15.

(b) By Exercise 5.14 we have

$$U(f+g) \leqslant U(f) + U(g) \qquad \text{and} \qquad L(f+g) \geqslant L(f) + L(g).$$

But both f and g are integrable on $[a, b]$, so

$$\int_a^b f(x)\,dx + \int_a^b g(x)\,dx = L(f) + L(g) \leqslant L(f+g)$$
$$\leqslant U(f+g) \leqslant U(f) + U(g) = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

We conclude that all the inequalities must in fact be equalities, therefore $f + g$ is integrable on $[a, b]$ and

$$\int_a^b \big(f(x) + g(x)\big)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$$

(c) From Exercise 5.14 we see that

$$U(\lambda f) = \lambda U(f) \qquad \text{and} \qquad L(\lambda f) = \lambda L(f) \qquad \text{if } \lambda \geqslant 0$$

and

$$U(\lambda f) = \lambda L(f) \qquad \text{and} \qquad L(\lambda f) = \lambda U(f) \qquad \text{if } \lambda < 0.$$

Since f is integrable on $[a, b]$ we have if $\lambda \geqslant 0$

$$\lambda \int_a^b f(x)\,dx = \lambda L(f) = L(\lambda f) \leqslant U(\lambda f) = \lambda U(f) = \lambda \int_a^b f(x)\,dx,$$

so the inequality is in fact an equality and λf is integrable with the claimed integral. If $\lambda < 0$ then

$$\lambda \int_a^b f(x)\,dx = \lambda U(f) = L(\lambda f) \leqslant U(\lambda f) = \lambda L(f) = \lambda \int_a^b f(x)\,dx$$

and we reach the same conclusion.

(e) This follows immediately from Exercise 5.14:

$$\int_a^b f(x)\,dx = U(f) \leqslant U(g) = \int_a^b g(x)\,dx.$$

Solution 5.16.

(a) Since f is integrable on $[a, b]$, it is bounded on $[a, b]$, so there exists an interval $[c, d]$ such that the image of f is a subset of $[c, d]$. Let $\varphi : \ [c, d] \longrightarrow \mathbf{R}$ be given by $\varphi(t) = |t|$, then φ is continuous, so by the given statement we conclude that $|f| = \varphi \circ f$ is integrable on $[a, b]$. For the inequality note that for all $x \in [a, b]$ we have

$$-|f(x)| \leqslant f(x) \leqslant |f(x)|,$$

so by Theorem 5.29:

$$-\int_a^b |f(x)|\,dx \leqslant \int_a^b f(x)\,dx \leqslant \int_a^b |f(x)|\,dx.$$

(b) Once again, the image of f is a subset of some interval $[c, d]$. Let $\varphi : \ [c, d] \longrightarrow \mathbf{R}$

be given by $\varphi(t) = t^2$ and apply the given statement together with Exercise 5.12.

(c) Since f and g are integrable, so are $f+g$, $(f+g)^2$, f^2, and g^2. Now use

$$fg = \frac{(f+g)^2 - f^2 - g^2}{2}.$$

(d) Easy induction argument using the previous part.

Solution 5.17.

(a) f is continuous, so it is integrable.

(b) g is continuous, so it is integrable.

(c) h is not integrable because the upper sums will always be given by by x^2, while the lower sums will always be given by by $-x^2$. This is similar to the rational indicator function.

(d) p is not integrable because it is unbounded: the definition of the Riemann integral requires the function be bounded.

(e) q is integrable because it is piecewise continuous and bounded.

(f) r is integrable because it is piecewise continuous and bounded.

Solution 5.18. Let $\varepsilon > 0$. Choose partition $P_\varepsilon = \{-1, 0, \varepsilon/2, 1\}$. Then

$$U(f, P_\varepsilon) = 0 \cdot (0 - (-1)) + 1 \cdot (\varepsilon/2 - 0) + 1 \cdot (1 - \varepsilon/2) = 1$$

and

$$L(f, P_\varepsilon) = 0 \cdot (0 - (-1)) + 0 \cdot (\varepsilon/2 - 0) + 1 \cdot (1 - \varepsilon/2) = 1 - \varepsilon/2$$

Hence $U(f, P_\varepsilon) - L(f, P_\varepsilon) = \varepsilon/2 < \varepsilon$. Therefore f is Riemann integrable.

Solution 5.19. Let $\varepsilon > 0$. Define a partition $P_\varepsilon = \{-1, -\varepsilon/3, \varepsilon/3, 1\}$. Then

$$U(f, P_\varepsilon) = 0 \cdot (-\varepsilon - (-1)) + 1 \cdot (\varepsilon/3 - (-\varepsilon/3)) + 0 \cdot (1 - \varepsilon) = 2\varepsilon/3,$$

and

$$L(f, P_\varepsilon) = 0 \cdot (-\varepsilon - (-1)) + 0 \cdot (\varepsilon/3 - (-\varepsilon/3)) + 0 \cdot (1 - \varepsilon) = 0.$$

Hence $U(f, P_\varepsilon) - L(f, P_\varepsilon) = 2\varepsilon/3 < \varepsilon$ and so f is Riemann integrable.

Solution 5.20.

(a) Improper, infinite interval of integration.

(b) Improper, infinite discontinuity at $\pi/2$.

(c) Proper.

(d) Improper, infinite discontinuity at $\frac{1}{3}$.

(e) Improper, infinite interval of integration.

(f) Improper, infinite discontinuity at 0.

(g) Proper.

(h) Improper, infinite discontinuity at $\frac{1}{2}$.

(i) Proper.

Solution 5.21.

(a) 2

(b) 6

(c) divergent

(d) divergent

(e) 1

(f) $\frac{\pi}{2}$

(g) $2\sqrt{5}$

(h) $\log 2$.