4 Limits of functions

Limits

Exercise 4.1. Using the ε-δ definition of limit (Definition 4.7), prove the following:

(a) $\lim_{x\to 3} 5x = 15$;

(b) $\lim_{x\to 3}(5x+2) = 17$;

(c) $\lim_{x\to 2} \dfrac{3x^2-12}{x-2} = 12$;

(d) $\lim_{x\to 3} \dfrac{3}{x} = 1$.

[**Hint**: This one is tricky! Notice that $\left|\dfrac{3}{x} - 1\right| \leqslant \dfrac{|x-3|}{|x|}$. What can you say about $\frac{1}{|x|}$ when $|x-3| < 1$?]

Exercise 4.2. Prove the following limits using the definition of limit at ∞ given in Definition 4.17:

(a) $\lim_{x\to\infty} c = c$ for each $c \in \mathbf{R}$.

(b) $\lim_{x\to\infty} \dfrac{1}{x^p} = 0$ for each $p > 0$.

(c) $\lim_{x\to\infty} \left(4 + \dfrac{1}{\sqrt{x}}\right) = 4$.

Exercise 4.3. Let $f: \mathbf{R} \longrightarrow \mathbf{R}$ be a function. We say that $\lim_{x\to\infty} f(x) = \infty$ if for every $r \in \mathbf{R}$ there exists $M \in \mathbf{R}$ such that if $x > M$ then $f(x) > r$.

(a) Prove that $\lim_{x\to\infty} \sqrt{x} = \infty$.

(b) Let f be a function such that $\lim_{x\to\infty} f(x) = \infty$.

Define the sequence (a_n) by $a_n = f(n)$ for all $n \in \mathbf{N}$. Prove that $\lim_{n\to\infty} a_n = \infty$.

What can you say if $\lim_{x\to\infty} f(x) = L$ for some $L \in \mathbf{R}$?

Weird functions

Exercise 4.4. Consider the function $f : \mathbf{R} \longrightarrow \mathbf{R}$ given by

$$f(x) = \begin{cases} 0 & x \in \mathbf{Q} \\ 1 & x \notin \mathbf{Q}. \end{cases}$$

Prove that $f(x)$ diverges as x approaches $\sqrt{2}$.
[**Hint**: Proceed by contradiction and consider $\varepsilon = 1/2$.]

Exercise 4.5. Consider the function $f : \mathbf{R} \longrightarrow \mathbf{R}$ given by

$$f(x) = \begin{cases} 0 & x \in \mathbf{Q} \\ x & x \notin \mathbf{Q}. \end{cases}$$

Prove that $f(x)$ converges as x approaches 0.

Exercise 4.6. Recall that each non-zero rational number $x \in \mathbf{Q}$ can be expressed as $x = p/q$ where p, q are relatively prime integers and $q > 0$. Consider *Thomae's function* $f : \mathbf{R} \longrightarrow \mathbf{R}$ given by:

$$f(x) = \begin{cases} 0 & x \notin \mathbf{Q} \\ 1 & x = 0 \\ \frac{1}{q} & x = \frac{p}{q} \neq 0. \end{cases}$$

(a) Show that for each $a \in \mathbf{R}$, $f(x)$ converges as x approaches a.

(b) Let $a \in \mathbf{R}$. Show that f is continuous at a if and only if $a \notin \mathbf{Q}$.

Continuity

Exercise 4.7. Use the definition of continuity (Definition 4.19) to prove that if two functions f, g are continuous at a, then so are the functions

(a) $f - g$

(b) $f + 2g$.

Exercise 4.8. Determine if the following functions are continuous on the interval $[-1, 1]$.

(a) $f(x) = |x|$;

(b) $f(x) = \begin{cases} (1-x)^{-1}, & x \neq 1; \\ 1, & x = 1; \end{cases}$

(c) $f(x) = \begin{cases} \dfrac{|x|}{x}, & x \neq 0; \\ 1, & x = 0. \end{cases}$

Exercise 4.9. Let $f\colon\ I \longrightarrow \mathbf{R}$ be continuous on the interval I. Prove the function g defined by $g(x) = |f(x)|$ is continuous on I.

Is the converse of the above statement True? If yes, prove it. If no, give a counterexample.

Exercise 4.10. Give a direct proof of the Composition and Continuity Theorem 4.27 (in other words, use directly the ε-δ definition of continuity).

Exercise 4.11. Let $f\colon\ [a,b] \longrightarrow \mathbf{R}$ be continuous. There exist $\delta_a, \delta_b > 0$ such that f is bounded on the closed intervals $[a, a+\delta_a]$ and $[b-\delta_b, b]$.

[**Hint**: Use an argument similar to the proof of Lemma 4.30.]

Exercise 4.12.

(a) Let $f\colon\ \mathbf{R} \longrightarrow \mathbf{R}$ be given by $f(x) = x^3 - 5x^2 + 7x - 9$. Prove that there is a real number c such that $f(c) = 100$.

(b) Let $f\colon\ \mathbf{R} \longrightarrow \mathbf{R}$ be given by $f(x) = x^5 - 3x^4 - 2x^3 - x + 1$. Prove that f has at least one root between 0 and 1.

(c) Show that the equation $x + \sin x = 1$ has at least one solution in the interval $[0, \pi/6]$.

(d) Let $f\colon\ [0,1] \longrightarrow [0,1]$ be a continuous function. Prove that there exists $c \in [0,1]$ such that $f(c) = c$.

[**Hint**: Define a function g so that $g(x) = 0$ exactly when $f(c) = c$.]

(e) Let $f\colon\ [0,1] \longrightarrow \mathbf{R}$ be a continuous function satisfying $f(0) = f(1) = 0$. Prove that there exists $c \in [0, 1/2)$ such that $f(c+1/2) = f(c)$.

Sequences of functions

Exercise 4.13. Let $f_0\colon\ \mathbf{R} \longrightarrow \mathbf{R}$ be the function defined by

$$f_0(x) = \begin{cases} 1+x & \text{if } -1 \leqslant x \leqslant 0, \\ 1-x & \text{if } 0 < x \leqslant 1, \\ 0 & \text{otherwise.} \end{cases}$$

For each positive integer n, define $f_n\colon\ \mathbf{R} \longrightarrow \mathbf{R}$ by

$$f_n(x) = f_0(x-n).$$

(a) Sketch the graphs of f_0, f_1, f_2, f_3 to get a feel for these functions.

(b) Find the image (range) of the function f_0.

(c) Prove that for all $n \in \mathbf{N}$, the function f_n is bounded on $\mathbf{R}$.

(d) Find the pointwise limit f of the sequence (f_n).

Exercise 4.14. For each $n \in \mathbf{N}$ define $f_n : [0,1] \longrightarrow \mathbf{R}$ by

$$f_n(x) = \frac{nx^2}{1+nx}.$$

(a) Prove that for all $n \in \mathbf{N}$, the function f_n is continuous.

(b) Find the pointwise limit f of the sequence $\big(f_n\big)$ and determine whether the sequence converges uniformly to f.

[**Hint**: You may need to use the following fact from Calculus: if g is a differentiable function and $g'(x) > 0$ for all $x \in [0,1]$, then g is an increasing function on $[0,1]$.]

Exercise 4.15. For $n \geqslant 1$, consider $f_n : \mathbf{R} \longrightarrow \mathbf{R}$ given by

$$f_n(x) = \frac{x^2+nx}{n}.$$

Find the pointwise limit f of the sequence (f_n) and show that f is continuous but that (f_n) does **not** converge uniformly to f.

Exercise 4.16. Let $E \subseteq \mathbf{R}$ and suppose (f_n) is a sequence of bounded functions $f_n : E \longrightarrow \mathbf{R}$, that is for each $n \in \mathbf{N}$ there exists $C_n \in \mathbf{R}$ such that

$$|f_n(x)| \leqslant C_n \qquad \text{for all } x \in E.$$

Suppose that the sequence (f_n) converges uniformly to a function $f : E \longrightarrow \mathbf{R}$. Prove that f is bounded.

Exercise 4.17. Let $E = [0,\infty)$. For each $n \in \mathbf{N}$, consider the function $f_n : E \longrightarrow \mathbf{R}$ defined by

$$f_n(x) = \frac{nx}{1+nx^2}.$$

(a) Fix $n \in \mathbf{N}$. Prove that f_n is bounded on E, more precisely that

$$\frac{nx}{1+nx^2} \leqslant \frac{\sqrt{n}}{2} \qquad \text{for all } x \in E.$$

(b) Find the pointwise limit f of the sequence (f_n).

(c) Show that f is not bounded on E.

(d) Is the convergence of (f_n) uniform?

Miscellaneous

Exercise 4.18. Let $f : [a, b] \longrightarrow \mathbf{R}$ be a continuous function.

(a) Suppose f is injective and $f(a) < f(b)$.

  i. Prove that $f(a) < f(x) < f(b)$ for all $x \in (a, b)$.

  ii. Prove that the image of f is the closed interval $[f(a), f(b)]$.

  iii. Prove that f is *strictly monotone increasing*:

$$\text{for all } x_1, x_2 \in [a, b] \text{ we have } x_1 < x_2 \quad \Rightarrow \quad f(x_1) < f(x_2).$$

(b) Suppose f is injective and $f(a) > f(b)$. Find the analogues of the statements in part (a) and convince yourself that the proofs work similarly.

(c) Prove that if $f : [a, b] \longrightarrow I$ is bijective, then $f^{-1} : I \longrightarrow [a, b]$ is continuous.

(A continuous bijective function whose inverse is also continuous is called a *homeomorphism*.)

Answers

Solution 4.1.

(a) Let $\varepsilon > 0$. Choose $\delta = \dfrac{\varepsilon}{5}$.

Assume $0 < |x-3| < \delta$.
Then
$$\begin{aligned} |5x-15| &= 5|x-3| \\ &< 5 \cdot \frac{\varepsilon}{5} \\ &= \varepsilon. \end{aligned}$$

Therefore $0 < |x-3| < \delta$ implies $|5x-15| < \varepsilon$.

(b) Let $\varepsilon > 0$. Choose $\delta = \dfrac{\varepsilon}{5}$.

Assume $0 < |x-3| < \delta$.
Then
$$\begin{aligned} |(5x+2)-17| &= 5|x-3| \\ &< 5 \cdot \frac{\varepsilon}{5} \\ &= \varepsilon. \end{aligned}$$

Therefore $0 < |x-3| < \delta$ implies $|(5x+2)-17| < \varepsilon$.

(c) Let $\varepsilon > 0$. Choose $\delta = \dfrac{\varepsilon}{3}$.

Assume $0 < |x-3| < \delta$.
Then
$$\begin{aligned} \left| \left(\frac{3x^2-12}{x-2} \right) - 12 \right| &= \left| \frac{3(x-2)(x+2)}{x-2} - 12 \right| \\ &= |3(x+2)-12| \\ &= |3x-6| = 3|x-2| \\ &< 3 \cdot \frac{\varepsilon}{3} \\ &= \varepsilon. \end{aligned}$$

Therefore $0 < |x-2| < \delta$ implies $\left| \left(\dfrac{3x^2-12}{x-2} \right) - 12 \right| < \varepsilon$.

(d) Let $\varepsilon > 0$. Choose $\delta = \mathsf{min}\{1, 2\varepsilon\}$. Then

i. $\delta \leqslant 2\varepsilon$

ii. $|x-3| < \delta \leqslant 1$. In particular, $x > 2$, and so $\dfrac{1}{|x|} < \dfrac{1}{2}$.

Assume $0 < |x-3| < \delta$.

Then
$$\begin{aligned} \left|\frac{3}{x} - 1\right| &\leqslant \frac{|x-3|}{|x|} \\ &= |x-3| \cdot \frac{1}{|x|} \\ &< \frac{1}{2}\delta \\ &\leqslant \varepsilon. \end{aligned}$$

Therefore $0 < |x-3| < \delta$ implies $\left|\left(\frac{3}{x}\right) - 1\right| < \varepsilon$.

Solution 4.2.

(a) Let $\varepsilon > 0$. Choose $M = 1$.

Then we have
$$|f(x) - L| = |c - c| = 0 < \varepsilon$$
for all $x > M$. So $\lim_{x \to \infty} c = c$.

(b) Let $\varepsilon > 0$. Choose $M = \varepsilon^{-1/p} > 0$ so $\frac{1}{M^p} = \varepsilon$.

Then we have
$$|f(x) - L| = \left|\frac{1}{x^p} - 0\right| = \frac{1}{x^p} < \frac{1}{M^p} = \varepsilon$$
for all $x > M > 0$. So $\lim_{x \to \infty} \frac{1}{x^p} = 0$.

(c) Let $\varepsilon > 0$. Choose $M = \frac{1}{\varepsilon^2} > 0$ so $\frac{1}{\sqrt{M}} = \varepsilon$.

Then
$$|f(x) - L| = \left|4 + \frac{1}{\sqrt{x}} - 4\right| = \frac{1}{\sqrt{x}} < \frac{1}{\sqrt{M}} = \varepsilon$$
for all $x > M > 0$. So $\lim_{x \to \infty} \left(4 + \frac{1}{\sqrt{x}}\right) = 4$.

Solution 4.3.

(a) Let $r \in \mathbf{R}$. Choose $M = r^2$.

Then we have
$$f(x) = \sqrt{x} > \sqrt{M} = \sqrt{r^2} \geqslant r$$
for all $x > M$. So $\lim_{x \to \infty} \sqrt{x} = \infty$.

(b) Let $r \in \mathbf{R}$. We know that $\lim_{x\to\infty} f(x) = \infty$, so there exists an $M' \in \mathbf{R}$ such that for all $x > M'$, $f(x) > r$. Pick $M = \lceil M' \rceil \geqslant M'$, then for all $n \in \mathbf{N}$:

$$n > M \Rightarrow n > M' \Rightarrow f(n) > r \Rightarrow a_n > r.$$

So indeed $\lim_{n\to\infty} a_n = \infty$.

The case that the limit is a real number is similar. Suppose $\lim_{x\to\infty} f(x) = L$. Let $\varepsilon > 0$. Then there exists an $M' \in \mathbf{R}$ such that for all $x > M'$, we have $|f(x) - L| < \varepsilon$. Again choose $M = \lceil M' \rceil \geqslant M'$. For all $n \in \mathbf{N}$, we have

$$n > M \Rightarrow n > M' \Rightarrow |f(n) - L| = |a_n - L| < \varepsilon.$$

So $\lim_{n\to\infty} a_n = L = \lim_{x\to\infty} f(x)$.

Solution 4.4. Suppose that f converges to $L \in \mathbf{R}$ as $x \longrightarrow \sqrt{2}$. Let $\varepsilon = 1/2$. Then there exists $\delta > 0$ such that for all $x \in \mathbf{R}$ such that $|x - \sqrt{2}| < \delta$, we have

$$|f(x) - L| < 1/2 \quad \Rightarrow \quad f(x) - \frac{1}{2} < L < f(x) + \frac{1}{2}.$$

By Corollary 2.61, there exists $x_0 \in \mathbf{Q}$ such that $|x_0 - \sqrt{2}| < \delta$. Plugging x_0 for x in the above inequalities for L we get:

$$-\frac{1}{2} = f(x_0) - \frac{1}{2} < L < f(x_0) + \frac{1}{2} = \frac{1}{2}.$$

On the other hand, letting $x_1 = \sqrt{2}$ we have that $x_1 \notin \mathbf{Q}$ and $|x_1 - \sqrt{2}| < \delta$. So plugging x_1 for x in the inequalities for L we get:

$$\frac{1}{2} = 1 - \frac{1}{2} = f(x_1) - \frac{1}{2} < L < f(x_1) + \frac{1}{2} = 1 + \frac{1}{2} = \frac{3}{2}.$$

So L must at the same time satisfy $L < \frac{1}{2}$ and $L > \frac{1}{2}$, contradiction.

Solution 4.5. Let $\varepsilon > 0$. Choose $\delta = \varepsilon$. If $x \in \mathbf{R}$ is such that $|x - 0| = |x| < \delta$, we have $|f(x)| \leqslant |x| < \delta = \varepsilon$. So $f(x)$ converges to 0 at 0.

Solution 4.6.

(a) We claim that $\lim_{x\to a} f(x) = 0$ for all $a \in \mathbf{R}$.

Let $a \in \mathbf{R}$. Let $\varepsilon > 0$. The challenge is to find $\delta > 0$ such that for each $x \in \mathbf{R}$

$$0 < |x - a| < \delta \quad \Rightarrow \quad |f(x) - 0| < \varepsilon.$$

By the Archimedean Principle, there exists $n \in \mathbf{N}$ such that $1/n < \varepsilon$. Given an integer q such that $1 \leqslant q \leqslant n$, let

$$\delta_q = \inf \left\{ \left| \frac{p}{q} - a \right| : \ p \in \mathbf{Z} \text{ and } \frac{p}{q} \neq a \right\}.$$

That way for each $p \in \mathbf{Z}$, either $p/q = a$ or $|p/q - a| \geqslant \delta_q$. To see that $0 < \delta_q < \infty$, by again applying the Archimedean Principle there exists $k \in \mathbf{Z}$ such that $k \leqslant qa < k+1$; that is, $k/q \leqslant a < (k+1)/q$. It follows that in the case $a = k/q$, $\delta_q = 1/q$. In the case $a \neq k/q$,

$$\delta_q = \min \left\{ a - \frac{k}{q}, \frac{k+1}{q} - a \right\} > 0.$$

Let $\delta = \min \{\delta_1, \delta_2, \ldots, \delta_n\} > 0$. Let $x \in \mathbf{R}$ such that $0 < |x - a| < \delta$. If $x \notin \mathbf{Q}$, then

$$|f(x) - 0| = |0 - 0| = 0 < \varepsilon.$$

Suppose that $x \in \mathbf{Q}$ and $x = p/q$, where p, q are relatively prime integers and $q > 0$. If $q \leqslant n$, then either $x = a$ or $|x - a| \geqslant \delta_q \geqslant \delta$, contrary to assumption. Hence we must have $q > n$. (This includes the case $x = 0$, as $0 = 0/1$ with $q = 1$.) Thus

$$|f(x) - 0| = |1/q - 0| = 1/q < 1/n < \varepsilon.$$

Therefore, if $0 < |x - a| < \delta$ then $|f(x) - 0| < \varepsilon$. That is, $\lim_{x \to a} f(x) = 0$.

(b) Let $a \in \mathbf{R}$. Since the domain of f is $\mathbf{R}$, f is continuous at a if and only if $\lim_{x \to a} f(x) = f(a)$. By Part (a), $\lim_{x \to a} f(x) = 0$. Hence f is continuous at a if and only if $f(a) = 0$ if and only if $a \notin \mathbf{Q}$.

Solution 4.7.

(a) Let $\varepsilon > 0$. Then $\varepsilon/2 > 0$. Since f and g are continuous at a, there exists $\delta_f, \delta_g > 0$ so that

- $|f(x) - f(a)| < \varepsilon/2$ whenever $x \in (a - \delta_f, a + \delta_f)$; and
- $|g(x) - g(a)| < \varepsilon/2$ whenever $x \in (a - \delta_g, a + \delta_g)$.

Let $\delta = \min \{\delta_f, \delta_g\}$. Using the Triangle Inequality, one has:

$$\begin{aligned} |(f(x) - g(x)) - (f(a) - g(a))| &= |(f(x) - f(a)) - (g(x) - g(a))| \\ &\leqslant |f(x) - f(a)| + |g(x) - g(a)| \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon \end{aligned}$$

whenever $x \in (a-\delta, a+\delta)$. So $f-g$ is continuous at a.

(b) Proceed similar to (a). Let $\varepsilon > 0$. Then $\varepsilon/3 > 0$. Since f and g are continuous at a, there exists $\delta_f, \delta_g > 0$ so that

- $|f(x) - f(a)| < \varepsilon/3$ whenever $x \in (a-\delta_f, a+\delta_f)$; and
- $|g(x) - g(a)| < \varepsilon/3$ whenever $x \in (a-\delta_g, a+\delta_g)$.

Let $\delta = \min\{\delta_f, \delta_g\}$. Using the Triangle Inequality, one has:

$$\begin{aligned} |(f(x)+2g(x)) - (f(a)+2g(a))| &= |(f(x)-f(a)) + (2g(x)-2g(a))| \\ &\leqslant |f(x)-f(a)| + 2|g(x)-g(a)| \\ &< \varepsilon/3 + 2\varepsilon/3 \\ &= \varepsilon \end{aligned}$$

whenever $x \in (a-\delta, a+\delta)$. So $f+2g$ is continuous at a.

Solution 4.8.

(a) Let $a \in [-1,1]$ and $\varepsilon > 0$. Choose $\delta = \varepsilon$.

Then for all $|x-a| < \delta$, one has

$$|f(x) - f(a)| = ||x| - |a|| \leqslant |x-a| < \delta = \varepsilon.$$

So $f(x) = |x|$ is continuous for all $a \in [-1,1]$.

Note that in this case, δ depends only on ε and not on the point a as well. This is *uniform continuity*, which is a stronger form of continuity.

(b) We show that f is discontinuous at 1. Let $\varepsilon = 1$ and $0 < \delta \in \mathbf{R}$. Then we may pick $x = \max\{1/2, 1-\delta/2\}$. Then $|x-1| = \min\{1/2, \delta/2\} \leqslant \delta/2 < \delta$. Moreover, as $0 < 1-x \leqslant 1/2 \leqslant x < 1$, we have

$$|f(x) - f(1)| = \left|\frac{1}{1-x} - 1\right| = \left|\frac{x}{1-x}\right| = \frac{x}{1-x} \geqslant 1 = \varepsilon.$$

So $f(x)$ is discontinuous at 1.

(c) We show that f is discontinuous at 0. Let $\varepsilon = 1$ and $0 < \delta \in \mathbf{R}$. Pick $x = -\delta/2$. Then

$$|f(x) - f(0)| = \left|\frac{|x|}{x} - 1\right| = \left|\frac{\delta/2}{-\delta/2} - 1\right| = |-1-1| = 2 > \varepsilon.$$

So $f(x)$ is discontinuous at 0.

Solution 4.9. Fix an arbitrary point $c \in I$ and $0 < \varepsilon$. By continuity of f at c there exists a $0 < \delta$ such that for all $x \in I$ such that $|x - c| < \delta$, we have

$$\Big| |f(x)| - |f(c)| \Big| \leqslant |f(x) - f(c)| \leqslant \varepsilon,$$

by the reverse triangle inequality Exercise 2.24. Hence $|f|$ is continuous at c.

The converse is not True. Consider the function:

$$f(x) = \begin{cases} 1 & x \notin \mathbf{Q} \\ -1 & x \in \mathbf{Q}, \end{cases}$$

then $|f|: \mathbf{R} \longrightarrow \mathbf{R}$ is the constant function 1 (therefore is continuous on $\mathbf{R}$), but f itself is not continuous at any $c \in \mathbf{R}$.

Solution 4.10. Let $f: A \longrightarrow \mathbf{R}$ be continuous at $a \in \mathbf{R}$ and $g: B \longrightarrow \mathbf{R}$ be continuous at $f(a)$, where $f(A) \subseteq B$. Let $\varepsilon > 0$. As g is continuous at $f(a)$, there exists a $\delta_g > 0$ such that for all $x \in B$,

$$|x - f(a)| < \delta_g \Rightarrow |g(x) - g(f(a))| < \varepsilon.$$

Now let $\varepsilon_f = \delta_g > 0$. As f is continuous at a, there exists a $\delta > 0$ such that for all $x \in A$,

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon_f.$$

But then,

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon_f = \delta_g \Rightarrow |g(f(x)) - g(f(a))| < \varepsilon,$$

for all $x \in A$, as $f(x) \in B$. Hence $g \circ f$ is continuous at a.

Solution 4.11. We give the proof of the existence of δ_a (the case of δ_b is very similar).

Let $\varepsilon = 1$. Then there exists a $\delta > 0$ such that for all $x \in [a, b]$,

$$|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon.$$

Choose $\delta_a = \min\{\delta, b - a\}$. So $\delta_a \leqslant \delta$ and $[a, a + \delta_a) \subseteq [a, b]$. Let $C := \max\{|f(a)| + 1, |f(a + \delta_a)|\}$.

Now let $x \in [a, a + \delta_a]$. If $x = a + \delta_a$. Then $|f(x)| = |f(a + \delta_a)| \leqslant C$. Otherwise $x \in [a, a + \delta)$. Then $|x - a| < \delta$, and

$$|f(x)| \leqslant |f(x) - f(a)| + |f(a)| < 1 + |f(a)| \leqslant C.$$

In any case, C is a bound for $f(x)$ on $[a, \delta_a]$.

Solution 4.12.

(a) Since f is a polynomial, f is continuous on $\mathbf{R}$.

Notice $f(0) = -9 < 100$ and $f(10) = 561 > 100$.

Therefore, by the Intermediate Value Theorem, there exists $c \in (0, 10)$ such that $f(c) = 100$.

(b) Find $f(0)$ and $f(1)$. Notice they have opposite signs.

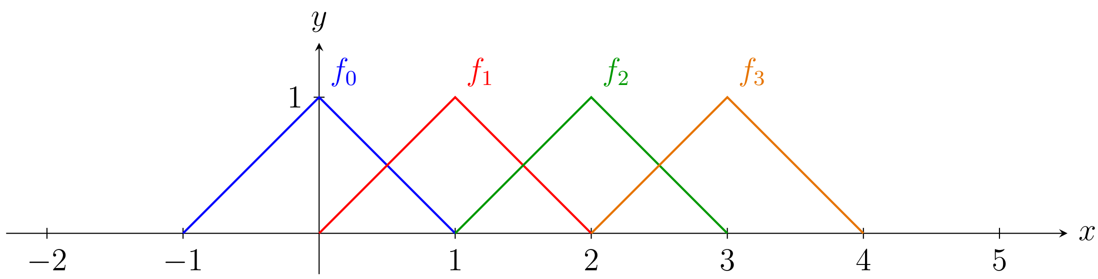
(c) Let $f(x) = x + \sin x - 1$. Compare the sign of this function at $x = 0$ and $x = \pi/6$.

(d) Let $g(x) = x - f(x)$. We have $g(0) = 0 - f(x) \leqslant 0$ and $g(1) = 1 - f(x) \geqslant 0$. Applying the Intermediate Value Theorem implies there exists $c \in [0, 1]$ so that $g(c) = 0$. Therefore $f(c) = c$.

(e) Proceed as in the previous part. Consider $g(x) = f(x + 1) + f(x)$.

Solution 4.13.

(a) Here are the graph sketches. The graphs continue along the x-axis outside of the triangles, but haven't been drawn in to avoid confusion.



(b) The image of f_0 is $[0, 1]$.

(c) By part (a), f_0 is bounded below by 0 and above by 1. By the definition of f_n, so is f_n for all $n \in \mathbf{N}$.

(d) Fix a real number x and let N be the smallest positive integer such that $x < N$. It follows from the definition of f_n that $f_n(x) = 0$ if $n > N$. Hence $\big(f_n(x)\big) \longrightarrow 0$ as $n \longrightarrow \infty$ and therefore $f : \mathbf{R} \longrightarrow \mathbf{R}$ is the constant zero function.

Solution 4.14.

(a) The function f_n is the quotient of two continuous functions, and the denominator $1 + nx$ is nonzero on $[0, 1]$, so f_n is continuous on $[0, 1]$.

(b) The pointwise limit is given by

$$f_n(x) = \frac{nx^2}{1+nx} = \frac{x^2}{\frac{1}{n}+x} \longrightarrow \frac{x^2}{0+x} = x \qquad \text{as } n \longrightarrow \infty,$$

so $f(x) = x$ for all $x \in [0,1]$.

For any $x \in [0,1]$ and any $n \in \mathbf{N}$ we have

$$|f_n(x) - f(x)| = \left| -\frac{x}{1+nx} \right| = \frac{x}{1+nx}.$$

Consider the function $g : \ [0,1] \longrightarrow \mathbf{R}$ given by $g(x) = x/(1+nx)$.

I claim that g is an increasing function on $[0,1]$:

$$g'(x) = \frac{1}{(1+nx)^2} > 0 \qquad \text{for all } x \in [0,1].$$

Therefore g attains its maximum at $x = 1$, which is $g(1) = 1/(1+n)$.

Going back to our sequence (f_n), we have

$$|f_n(x) - f(x)| = \frac{x}{1+nx} \leqslant \frac{1}{1+n}.$$

Now let $\varepsilon > 0$. Take $M = \lceil \frac{1}{\varepsilon} - 1 \rceil$, then if $n > M$ then $\frac{1}{1+n} < \varepsilon$. Therefore, for any $x \in [0,1]$ we have that if $n > M$ then

$$|f_n(x) - f(x)| \leqslant \frac{1}{1+n} < \varepsilon.$$

Since M depends only on ε and not on x, the convergence is uniform.

Solution 4.15. The sequence (f_n) converges pointwise to $f : \ \mathbf{R} \longrightarrow \mathbf{R}$ given by $f(x) = x$:

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x^2+nx}{n} = (x^2) \lim_{n\to\infty} \frac{1}{n} + \lim_{n\to\infty} x = 0 + x = x.$$

We have seen that the function f is continuous.

However, we have

$$|f_n(x) - f(x)| = \left| \frac{x^2+nx}{n} - x \right| = \frac{x^2}{n}.$$

For a fixed $x \in \mathbf{R}$ and $\varepsilon > 0$, we can take $N > x^2/\varepsilon$ to deduce pointwise convergence at x.

But to do so uniformly over all $x \in \mathbf{R}$ we would require N to satisfy $N > x^2/\varepsilon$ for all $x \in \mathbf{R}$, which is clearly impossible.

Solution 4.16. Take $\varepsilon = 1$ and let $M \in \mathbf{N}$ be such that for all $n \in \mathbf{N}$:

$$\text{if } n > M \text{, then } |f_n(x) - f(x)| < \varepsilon = 1 \quad \text{for all } x \in E.$$

In particular, if we take $n = M + 1$ we have

$$\begin{aligned} |f(x)| &= |f(x) - f_{M+1}(x) + f_{M+1}(x)| \\ &\leqslant |f(x) - f_{M+1}(x)| + |f_{M+1}(x)| \\ &\leqslant 1 + C_{M+1} \qquad \text{for all } x \in E. \end{aligned}$$

We conclude that f is bounded.

Solution 4.17.

(a) For any $x \in E$ we have

$$0 \leqslant (1 - x\sqrt{n})^2 \Rightarrow 0 \leqslant 1 - 2x\sqrt{n} + nx^2 \Rightarrow 2x\sqrt{n} \leqslant 1 + nx^2 \Rightarrow \frac{nx}{1+nx^2} \leqslant \frac{\sqrt{n}}{2}.$$

(b) Now we fix $x \in E$. If $x = 0$ then $f_n(x) = 0$ for all $n \in \mathbf{N}$, so $f(0) = 0$.

If $x > 0$ we have

$$\lim_{n\to\infty} \frac{nx}{1+nx^2} = \lim_{n\to\infty} \frac{x}{\frac{1}{n} + x^2} = \frac{x}{0 + x^2} = \frac{1}{x}.$$

So for $x > 0$ we have $f(x) = \frac{1}{x}$.

(c) For any $C \in \mathbf{R}$, there exists (by the Archimedean Principle) $n \in \mathbf{N}$ such that $n > C$. Let $x = 1/n$, then $f(x) = n > C$. So f is not bounded on E.

(d) No, by Exercise 4.16.

Solution 4.18.

(a) i. Suppose there exists $x \in (a, b)$ such that $f(x) \leqslant f(a) < f(b)$. Since f is injective, this is in fact $f(x) < f(a) < f(b)$. Apply the Intermediate Value Theorem to f on the interval $[x, b]$: since $y = f(a) \in \mathbf{R}$ is strictly between $f(x)$ and $f(b)$, there exists $c \in (x, b)$ such that $f(c) = y = f(a)$. But $a < x < c < b$, so $a \neq c$, contradicting the injectivity of f.

The other possibility is that there exists $x \in (a, b)$ such that $f(a) < f(b) \leqslant f(x)$, disproved similarly.

ii. By the previous part we have $\operatorname{im}(f) \subseteq [f(a), f(b)]$.

For the other inclusion, let $y \in [f(a), f(b)]$. If $y = f(a)$ or $y = f(b)$, then $f \in \operatorname{im}(f)$. Otherwise, y is strictly between $f(a)$ and $f(b)$, so we conclude that y is in the image of f by using the Intermediate Value Theorem.

iii. Suppose f is not strictly monotone increasing, so there exist $x_1, x_2 \in [a, b]$ such that $x_1 < x_2$ but $f(x_1) \geqslant f(x_2)$. Since f is injective, this is in fact $f(x_1) > f(x_2)$. By the previous two parts, we have

$$f(a) < f(x_2) < f(x_1) < f(b).$$

Now apply the Intermediate Value Theorem to f on the interval $[a, x_1]$: since $y = f(x_2)$ is strictly between $f(a)$ and $f(x_1)$, there exists $c \in (a, x_1)$ such that $f(c) = y = f(x_2)$, but $a < c < x_1 < x_2 < b$, so $c \neq x_2$, contradicting the injectivity of f.

(b) The statements are:

- $f(b) < f(x) < f(a)$ for all $x \in (a, b)$.
- The image of f is the closed interval $[f(b), f(a)]$.
- f is *strictly monotone decreasing*:

$$\text{for all } x_1, x_2 \in [a, b] \text{ we have } x_1 < x_2 \quad \Rightarrow \quad f(x_1) > f(x_2).$$

(c) From the previous parts, we know that f is strictly monotone, say strictly monotone increasing. (The other case is similar.)

Let $y \in I$ and let (y_n) be a sequence in I such that $y_n \longrightarrow y$. Take $x = f^{-1}(y)$ and $x_n = f^{-1}(y_n)$ for all $n \in \mathbf{N}$. Note that $x_n \in [a, b]$, so the sequence (x_n) is bounded.

We want to show that $x_n \longrightarrow x$. Suppose this is not True, then there exists $\varepsilon > 0$ such that for all $M \in \mathbf{N}$, there exists $n > M$ such that $|x_n - x| \geqslant \varepsilon$. In particular, for $M = 0$ there exists $n_0 > 0$ such that $|x_{n_0} - x| \geqslant \varepsilon$. Then for $M = n_0$ there exists $n_1 > n_0$ such that $|x_{n_1} - x| \geqslant \varepsilon$. For $M = n_1$ there exists $n_2 > n_1$ such that $|x_{n_2} - x| \geqslant \varepsilon$.

Continuing in this manner, we get a subsequence (x_{n_k}) of (x_n) such that $|x_{n_k} - x| \geqslant \varepsilon$ for all $k \in \mathbf{N}$. However, (x_{n_k}) is bounded since (x_n) is bounded, so by the Bolzano–Weierstrass Theorem there is a subsequence that converges to some $x' \in [a, b]$. Since $|x_{n_k} - x| \geqslant \varepsilon$ for all $k \in \mathbf{N}$, $x' \neq x$. But f is continuous, so $f(x') = y = f(x)$, contradicting the injectivity of f.

Hence $x_n \longrightarrow x$, which implies that f^{-1} is continuous at y.