1 Mathematical logic and proof

Logic and notation

Exercise 1.1. Write the following statements as a conditional statement in the form $p \Rightarrow q$.

(a) "A monkey is happy only if he is eating a banana."

(b) "A snake will not bite you provided you don't step on its tail."

(c) "A donkey laughs whenever he sees a mule."

(d) "Happiness is a necessary condition for Wealth."

(e) "Happiness is a sufficient condition for Wealth."

Exercise 1.2. Indicate whether each statement is True or False.

(a) Jupiter is a planet and Neptune is a moon.

(b) Jupiter is a planet or Neptune is a moon.

(c) Elvis was a woman or Cleopatra was a man.

(d) *TikTok* is a social media app, or *Lord of the Rings* was written by J.R.R. Tolkien.

(e) If the capital of Egypt is Cairo, then apples can be used to make cider.

(f) If Napolean was born in Zimbabwe, then the eigenstates of the quantum harmonic oscillator are proportional to Hermite polynomials.

(g) If the dodo is extinct, then pigs can fly!

(h) It is not the case that if Luke Skywalker was a Jedi, then his father was not Darth Vader.

Exercise 1.3. Translate the following into mathematical notation.

(a) Six is not prime or eleven is not prime.

(b) The square of 10 is 50 and the cube of 5 is 12.

(c) If 7 is an integer then 6 is not an integer.

(d) If both 2 and 5 are prime then 2×5 is not prime.

Which of these are True, which are False, and which are neither?

Exercise 1.4. Construct truth tables for the following statements:

(a) $(p \wedge q) \vee (\neg p \wedge \neg q)$.

(b) $[\neg q \wedge (p \Rightarrow q)] \Rightarrow \neg p$.

(c) $[(p \vee q) \wedge r] \Rightarrow (p \wedge r)$.

Which of the statements above is logically equivalent to a connective we have studied? Which of the statements above is a tautology?

Exercise 1.5. Translate the following into mathematical notation.

(a) All rational numbers are larger than 6.

(b) There is a real number solution to $x^2 + 3x - 7 = 0$.

(c) There is a natural number whose cube is 8.

Which of these are True, which are False, and which are neither?

Exercise 1.6. Translate the following mathematical statements into English:

(a) $(\forall a \in \mathbf{Q})\ a + 0 = a$.

(b) $(\forall x \in \mathbf{R})\ x^2 > 1$.

Exercise 1.7. The following two statements look similar, but say very different things. Which is True, and which is False?

(a) $(\exists b \in \mathbf{Z})\ [(\forall a \in \mathbf{Z})\ \ a + b = 0]$;

(b) $(\forall a \in \mathbf{Z})\ [(\exists b \in \mathbf{Z})\ \ a + b = 0]$.

Exercise 1.8. Find the negation of

(a) $(\forall x \in \mathbf{R})\ \ x^2 = 10$;

(b) $(\exists y \in \mathbf{N})\ \ y < 0$;

(c) $(\exists a \in \mathbf{N})\ [(\forall x \in \mathbf{R})\ \ ax = 4]$;

(d) $(\forall y \in \mathbf{Q})\ [(\exists x \in \mathbf{R})\ \ x/y = 30]$.

Exercise 1.9. Verify the following:

(a) $p \wedge q \Leftrightarrow q \wedge p$;

(b) $\neg(p \Rightarrow q) \Leftrightarrow (p \wedge \neg q)$

What can you conclude about (i) $p \wedge q \Leftrightarrow q \wedge p$ and (ii) $\neg(p \Rightarrow q) \Leftrightarrow (p \wedge \neg q)$?

Proof techniques

Exercise 1.10. In each of the following cases, use a counterexample to show that the statement is False.

(a) If the product of two integers is even then both of those integers are even.

(b) For all real numbers, if $x^2 = y^2$ then $x = y$.

(c) Let $a \in \mathbf{Z}$. If a divided by 7 gives remainder of 4, then $5a$ divided by 7 gives remainder of 4.

Exercise 1.11. Prove the following results involving integers.

(a) The product of an even integer with an odd integer is even.

(b) The sum of an even integer and an odd integer is odd.

(c) The cube of an odd integer is odd.

(d) Let $a \in \mathbf{Z}$. If a divided by 7 gives remainder of 4, then $15a$ divided by 7 gives remainder of 4.

(e) If k is odd, then $k^2 - 1$ is divisible by 4.

Exercise 1.12. For each equation below, list the sets of numbers ($\mathbf{N}, \mathbf{Z}, \mathbf{Q}, \mathbf{R}$) in which there exists at least one solution.

(a) $x^2 - 8 = 0$

(b) $x + 8 = 0$

(c) $4x = 8$

(d) $x^2 + 8 = 0$

(e) $3x + 8 = 0$

(f) $0x = 8$.

Exercise 1.13. Prove the following theorems using the contrapositive.

(a) Let $n \in \mathbf{Z}$. If n^4 is even, then n is even.

(b) Let $n \in \mathbf{Z}$. If n^3 is odd, then n is odd.

(c) For all $m, n \in \mathbf{Z}$, if mn is odd then m and n are odd.

Exercise 1.14. Consider the following statement:

"Let $p, q \in \mathbf{Z}$ with $p, q > 0$. If $pq = 1$ then $p = q = 1$."

Prove this statement by contradiction: suppose $pq = 1$ but at least one of p and q is not equal to 1, and arrive at a contradiction.

Exercise 1.15. Consider the following equation

$$x^2 - n^2y^2 = 1,$$

where $n \in \mathbf{Z}_{\geqslant 1}$ is fixed.

Show that the equation has no solutions $x, y \in \mathbf{Z}_{\geqslant 1}$ using a proof by contradiction via the following steps:

- Begin by assuming that there exists a positive integer solution.
- Factor the left-hand side, and conclude that both factors are integers.
- Using the result of Exercise 1.14, conclude both factors equal 1.
- Attempt to solve the two equations you have just developed and arrive at a contradiction.

Exercise 1.16 (The Adventures of π-casso, the Mathematical Artist)**.** Your friend π-casso has invited you over to see their new artwork. Before they unveil it, they tell you that it is a 3×3 grid of squares, each painted either red or blue. They go on to say that they used a special rule to paint it:

Every square has either 2 or 4 blue neighbours.

(a) Suppose that they tell you the centre square is red. Prove that the remaining squares are all blue using a contradiction argument. [**Hint**: You will need to consider 2 cases.]

(b) Suppose that they tell you the centre square is blue. Prove that at least one of the remaining squares is red using a contradiction argument.

(c) How many possible paintings are there that satisfy the special rule?

(d) **Extension.** Consider the paintings of 4×4 grids that use the same special rule. What are the possibilities?

Exercise 1.17. Prove the following theorems by dividing into two or more cases.

(a) Let $n \in \mathbf{Z}$. Prove that if n is not divisible by 3, then n^2 is not divisible by 3.

(b) For all $m, n \in \mathbf{Z}$, if m and n are either both odd or both even then $m + n$ is even.

Exercise 1.18. Prove that the following numbers are irrational:

(a) $\sqrt{3}$ (b) $\sqrt{15}+\sqrt{5}$ (c) $\log_2(7)$.

Exercise 1.19. Prove by induction that each formula is True for every integer $n \geqslant 1$.

(a) $2+7+12+\cdots+(5n-3)=\frac{1}{2}n(5n-1)$;

(b) $1+2\cdot 2+3\cdot 2^2+4\cdot 2^3+\cdots+n\cdot 2^{n-1}=1+(n-1)\,2^n$;

(c) $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\dfrac{1}{3\cdot 4}+\cdots+\dfrac{1}{n(n+1)}=\dfrac{n}{n+1}$;

(d) $\dfrac{a^{n+1}-b^{n+1}}{a-b}=a^n+a^{n-1}b+a^{n-2}b^2+\cdots+ab^{n-1}+b^n \quad (a\neq b)$.

Begin by rewriting these equations using summation notation.

Exercise 1.20. Prove by induction that the following statements are True for every natural number n:

(a) 3 is a factor of n^3-n+3;

(b) 9 is a factor of $10^{n+1}+3\cdot 10^n+5$;

(c) 4 is a factor of 5^n-1;

(d) the polynomial $x-y$ is a factor of the polynomial x^n-y^n;

(e) $7^{2n}-48n-1$ is divisible by 2304 (for all $n\geqslant 1$).

Exercise 1.21. Write the following inequalities in summation notation and then prove them, using summation notation throughout your proof. If you find this difficult, first try the proofs using more informal notation.

(a) $1^3+2^3+\cdots+(n-1)^3<\frac{1}{4}n^4<1^3+2^3+\cdots+n^3$ for all $n\geqslant 2$;

(b) $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots+\frac{1}{\sqrt{n}}\geqslant\sqrt{n}$ for all $n\geqslant 1$.

Exercise 1.22. In each case try to find $n_0\in\mathbf{N}$ such that the inequality appears to be True for all $n\geqslant n_0$.

If you think you have found such an n_0, give a proof by induction that the inequality is True for all $n\geqslant n_0$.

If you think that such n_0 cannot exist, try to prove this.

(a) $1+2n\leqslant 3^n$;

(b) $n!>2^n$;

(c) $n(n+1)\geqslant(2n-1)^2$;

(d) $n! > 2n^3$.

Exercise 1.23. Prove that for every natural number n we have

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & 0 \\ n & 1 \end{bmatrix}.$$

Answers

Solution 1.1.

(a) "A monkey is happy" $\Rightarrow$ "The monkey is eating a banana."

(b) "You don't step on the tail of a snake" $\Rightarrow$ "The snake will not bite you."

(c) "A donkey sees a mule" $\Rightarrow$ "The donkey laughs at him."

(d) Wealth $\Rightarrow$ Happiness.

(e) Happiness $\Rightarrow$ Wealth.

Solution 1.2.

(a) False: Neptune is a planet, not a moon.

(b) True: Jupiter **is** a planet.

(c) False: both parts are false.

(d) True: at least one of the parts is true (in this case, both are true).

(e) True.

(f) True: Napolean was born in Corsica, France. The second part happens to be true, although it doesn't matter in this case.

(g) False: Dodos **are** unfortunately extinct (hunted to extinction by man), but pigs cannot fly.

(h) True.

Solution 1.3. Let P = the set of prime numbers.

(a) $(\neg(6 \in P)) \vee (\neg(11 \in P))$.

(b) $(10^2 = 50) \wedge (5^3 = 12)$.

(c) $(7 \in \mathbf{Z}) \Rightarrow (\neg(6 \in \mathbf{Z}))$.

(d) $((2 \in P) \wedge (5 \in P)) \Rightarrow (\neg(2 \times 5 \in P))$.

Solution 1.4.

(a)

p	q	$(p \wedge q) \vee (\neg p \wedge \neg q)$
T	T	T
T	F	F
F	T	F
F	F	T

This is logically equivalent to the biconditional connective.

(b)

p	q	$[\neg q \wedge (p \Rightarrow q)] \Rightarrow \neg p$
T	T	T
T	F	T
F	T	T
F	F	T

This is a tautology.

(c)

p	q	r	$[(p \vee q) \wedge r] \Rightarrow (p \wedge r)$
T	T	T	T
T	T	F	T
T	F	T	T
T	F	F	T
F	T	T	F
F	T	F	T
F	F	T	T
F	F	F	T

Solution 1.5.

(a) $(\forall r \in \mathbf{Q})\ \ r > 6$; False.

(b) $(\exists x \in \mathbf{R})\ \ x^2 + 3x - 7 = 0$; True.

(c) $(\exists n \in \mathbf{N})\ \ n^3 = 8$; True.

Solution 1.6. There are of course many answers.

(a) "Adding zero to any rational number doesn't change the number."

(b) "The square of any real number is greater than one."

Solution 1.7. The second statement is True, the first statement is False.

Solution 1.8.

(a) $(\exists x \in \mathbf{R})\ \ x^2 \neq 10$;

(b) $(\forall y \in \mathbf{N})\ \ y \geqslant 0$;

(c) $(\forall a \in \mathbf{N})\ [(\exists x \in \mathbf{R})\ \ ax \neq 4]$;

(d) $(\exists y \in \mathbf{Q})\ [(\forall x \in \mathbf{R})\ \ x/y \neq 30]$.

Solution 1.9.

(a)

p	q	$p \wedge q$	$q \wedge p$	$p \wedge q \Leftrightarrow q \wedge p$
T	T	T	T	T
T	F	F	F	T
F	T	F	F	T
F	F	F	F	T

As this is a tautology, the two statements $p \wedge q$ and $q \wedge p$ are logically equivalent.

(b)

p	q	$\neg(p \Rightarrow q)$	$p \wedge \neg q$	$\neg(p \Rightarrow q) \Leftrightarrow (p \wedge \neg q)$
T	T	F	F	T
T	F	T	T	T
F	T	F	F	T
F	F	F	F	T

As this is a tautology, the two statements $\neg(p \Rightarrow q)$ and $p \wedge \neg q$ are logically equivalent.

Solution 1.10.

(a) Consider $3 \times 4 = 12$ which is even, but 3 is odd, so the statement is False.

(b) Put $x = 1$ and $y = -1$. Then $x^2 = y^2$, but $x \neq y$.

(c) Put $a = 11$, so $a = 7 + 4$ and the remainder is 4. But $5a = 55 = 7 \times 7 + 6$ and the remainder is $6 \neq 4$.

Solution 1.11. These proofs will all follow the same basic pattern. Identify the premise and express it in mathematical notation; identify the conclusion and express it in mathematical notation; connect the two with a reasoned argument.

(a) **Proof:**

1. Let x be an even integer. (premise)
2. There exists $k \in \mathbf{Z}$ such that $x = 2k$ (1, defn of even)
3. Let y be an odd integer. (premise)
4. There exists $\ell \in \mathbf{Z}$ such that $y = 2\ell + 1$ (3, defn of odd)
5. $xy = (2k)(2\ell + 1)$ (2,4)
6. $xy = 2(2k\ell + k)$ (algebra)

7. xy is even (defn of even, 6)
8. If x is even and y is odd, then xy is even. (1,3,7)

(b) **Proof:**

1. Let x be an even integer. (premise)
2. There exists $k \in \mathbf{Z}$ such that $x = 2k$ (1, defn of even)
3. Let y be an odd integer. (premise)
4. There exists $\ell \in \mathbf{Z}$ such that $y = 2\ell + 1$ (3, defn of odd)
5. $x + y = (2k) + (2l + 1)$ (2,4)
6. $x + y = 2(k + l) + 1$ (algebra)
7. $x + y$ is odd (defn of odd, 6)
8. If x is even and y is odd, then $x + y$ is odd. (1,3,7)

(c) **Proof:**

1. Let x be an odd integer. (premise)
2. There exists $k \in \mathbf{Z}$ such that $x = 2k + 1$ (1, defn of odd)
3. $x^3 = (2k + 1)^3$ (2)
4. $x^3 = 8k^3 + 12k^2 + 6k + 1$ (algebra)
5. $x^3 = 2(4k^3 + 6k^2 + 3k) + 1$ (algebra)
6. x^3 is odd (5, defn of odd)
7. If x is odd, then x^3 is odd (1,6)

(d) **Proof:**

1. a divided by 7 gives a reminder of 4. (premise)
2. $a \equiv 4 \mod 7$ (1, defn of reminder)
3. $15a = 7(2a) + a$ (algebra)
4. $15a \equiv a \equiv 4 \mod 7$ (2,3)
5. $15a$ divided by 7 gives a reminder of 4. (4)
6. If a divided by 7 gives a reminder of 4, then $15a$ divided by 7 gives a reminder of 4. (1,5)

(e) **Proof:**

1. Let k be an odd integer. (premise)
2. There exists $n \in \mathbf{Z}$ such that $k = 2n + 1$ (1, defn of odd)
3. $k^2 - 1 = (2n + 1)^2 - 1$ (2)

4. $k^2 - 1 = 4n^2 + 4n + 1 - 1$ (algebra)
5. $k^2 - 1 = 4(n^2 + n)$ (algebra)
6. $k^2 - 1$ is divisible by 4 (5)
7. If k is an odd integer, then $k^2 - 1$ is divisible by 4 (1,6)

Solution 1.12. The sets that go with each equation are:

(a) $\mathbf{R}$
(b) $\mathbf{Z}, \mathbf{Q}, \mathbf{R}$
(c) $\mathbf{N}, \mathbf{Z}, \mathbf{Q}, \mathbf{R}$
(d) none
(e) $\mathbf{Q}, \mathbf{R}$
(f) none.

Solution 1.13.

(a) We prove the contrapositive: if n is odd then n^4 is odd.

Assume n is odd. Then there exists $k \in \mathbf{Z}$ such that $n = 2k + 1$. We compute

$$(2k+1)^4 = 16k^4 + 32k^3 + 24k^2 + 8k + 1 = 2(8k^4 + 16k^3 + 12k^2 + 4k) + 1.$$

Therefore, n^4 is odd.

(b) We prove the contrapositive: if n is even then n^3 is even.

Assume m is even. Then $m = 2k$ for some $k \in \mathbf{Z}$. Computing $m^3 = (2k)^3 = 8k^3 = 2(4k^3)$ gives that m^3 is even.

(c) We prove the contrapositive: if m is even or n is even, then mn is even.

Without loss of generality, assume that m is even (otherwise interchange the roles of m and n). Then $m = 2k$ for some $k \in \mathbf{Z}$, so $mn = 2kn$, which is even.

Solution 1.14. We proceed by contradiction.

Suppose that $pq = 1$ but that the statement $p = q = 1$ is False. In other words, we have $p > 1$ or $q > 1$.

Without loss of generality, $p > 1$ (if it is the other way around, just interchange p and q in the argument).

Since $q \in \mathbf{Z}$ and $q > 0$, we have $q \geqslant 1$.

But $p > 1$ and $q \geqslant 1$ implies that $p \cdot q > 1 \cdot 1$, that is $pq > 1$, contradicting $pq = 1$.

Solution 1.15. We proceed by contradiction.

Suppose the equation $x^2 - n^2y^2 = 1$ has a solution with $x, y \in \mathbf{Z}_{\geqslant 1}$. We have then

$$(x - ny)(x + ny) = 1.$$

Both factors on the left hand side are certainly integers. In fact, they are both positive, which is obvious for $x + ny$ and then follows for $x - ny$ since the product has to be 1.

We are therefore in the situation of Exercise 1.14 with $p = x - ny$ and $q = x + ny$. From there we conclude that $x - ny = x + ny = 1$. But this implies that

$$0 = 1 - 1 = (x + ny) - (x - ny) = 2ny,$$

contradicting the fact that $ny \geqslant 1$ since both $n, y \geqslant 1$.

Solution 1.16.

(a) We proceed by contradiction. Suppose that another square besides the central square is red. Then it must be an edge square or a corner square.

- If a corner square is red, then the edge squares adjacent to it only have at most one blue neighbour, as the central and one corner are both red.
- If an edge square is red, then the corner squares adjacent to it have at most one blue neighbour, as they only have two neighbours.

So both cases lead to a contradiction.

(b) Again, we proceed by contradiction. Suppose that all squares are blue. Then the edge squares have three blue neighbours, which is against the rule. So there is at least one red square.

(c) Since the corner squares only have two edge square neighbours, all edge squares must be blue. Each edge square has three neighbours, so it must be the case that each edge square is neighbour to exactly one red square. If this square is central, then all corners must be blue and we get the configuration from part (a). If this square is a corner, then its adjacent corners must be blue, and the centre must be blue, hence the corner opposite must be red. This produces two configurations by considering the two sets of opposite corners.

So in total, there are 3 possible paintings.

(Random note: a mathematician would be much more likely to count this as "2 possible paintings", because the 2 with opposite red corners are the same up to symmetry. Which tells you that, unfortunately, a mathematician has a 50% chance of hanging the picture upside down . . .)

(d) Again, the edge squares must be blue to accommodate for the corners. The difference is that in this case, there are four pairs of edge squares that share disjoint sets of one centre and one corner square, and exactly one of them must be blue. Because of this pairing, we do not need to consider corner squares, as their colour is determined by the central square in its pair.

Note that each central square already has two blue edge neighbours, so it must be adjacent to two red squares or two blue squares. This gives us 4 possibilities. One where all central squares are red, one where all central squares are blue, and two where the central squares are in a chequerboard pattern.

(Would you believe that there are no possible paintings for a 5×5 grid? Check to see if it's true.)

Solution 1.17.

(a) Let $n \in \mathbf{Z}$. The remainder of the division of n by 3 is either 0, 1, or 2.

If n is not divisible by 3, then the remainder is either 1 or 2.

- If the remainder is 1, then there exists $k \in \mathbf{Z}$ such that $n = 3k + 1$. Then

$$n^2 = (3k+1)^2 = 9k^2 + 6k + 1 = 3(3k^2 + 2k) + 1,$$

so n^2 is not divisible by 3.

- If the remainder is 2, then there exists $k \in \mathbf{Z}$ such that $n = 3k + 2$. Then

$$n^2 = (3k+2)^2 = 9k^2 + 12k + 4 = 3(3k^2 + 4k + 1) + 1,$$

so n^2 is not divisible by 3.

(b) The case where both m and n are even is Lemma 1.39 in the lecture slides. So it remains to consider the case where both m and n are odd. There exist $k, \ell \in \mathbf{Z}$ such that $m = 2k + 1$ and $n = 2\ell + 1$. Then

$$m + n = 2k + 1 + 2\ell + 1 = 2k + 2\ell + 2 = 2(k + \ell + 1),$$

which is even.

Solution 1.18.

(a) We proceed by contradiction.

Suppose there exist integers p and q with no common factors such that $(p/q)^2 = 3$.

Then $p^2 = 3q^2$, so p^2 is divisible by 3. By Exercise 1.17 (a), this means that p is divisible by 3. So there exists $r \in \mathbf{Z}$ such that $p = 3r$.

Then $(3r)^2 = 3q^2$, so $3r^2 = q^2$, so q^2 is divisible by 3. Again by Exercise 1.17 (a), this means that q is divisible by 3.

Hence 3 is a common factor of p and q, contradicting the fact that they have no common factors.

(b) We proceed by contradiction.

Let $r = \sqrt{15} + \sqrt{5}$ and suppose r is rational.

We have

$$r^2 = 15 + 10\sqrt{3} + 5 \quad \Rightarrow \quad 10\sqrt{3} = r^2 - 20 \quad \Rightarrow \quad \sqrt{3} = \frac{r^2}{10} - 2.$$

But if $r \in \mathbf{Q}$ then $(r^2/10) - 2 \in \mathbf{Q}$, so $\sqrt{3} \in \mathbf{Q}$, contradicting Part (a).

(c) We proceed by contradiction.

Suppose there exist $p, q \in \mathbf{Z}$ with $q \geqslant 1$ such that $\log_2(7) = p/q$. Then $2^{p/q} = 7$, so $2^p = 7^q$.

Since $q \geqslant 1$ we have $7^q \geqslant 7$. So $p \geqslant 1$ (otherwise $p \leqslant 0$ so $2^p \leqslant 1$). Now note that 7^q is odd for all $q \geqslant 1$, while 2^p is even for all $p \geqslant 1$, so $2^p = 7^q$ is impossible.

Solution 1.19.

(a) In summation notation:

$$\sum_{k=1}^{n} (5k - 3) = \frac{1}{2} n(5n - 1).$$

Base case $n = 1$: $2 = 2$, True.

Induction step: let $n \geqslant 1$ be arbitrary but fixed and suppose that the formula holds for n. We have

$$\sum_{k=1}^{n+1} (5k-3) = \left(\sum_{k=1}^{n} (5k - 3) \right) + 5(n+1) - 3 = \frac{n(5n-1)}{2} + 5n + 2 = \frac{(n+1)(5n+4)}{2},$$

so the formula holds for $n + 1$.

(b) In summation notation:

$$\sum_{k=1}^{n} k2^{k-1} = 1 + (n - 1)2^n.$$

(c) In summation notation:

$$\sum_{k=1}^{n} \frac{1}{k(k+1)} = \frac{n}{n+1}.$$

(d) In summation notation:

$$\frac{a^{n+1} - b^{n+1}}{a - b} = \sum_{k=0}^{n} a^{n-k} b^k.$$

For the proof, it's better to rewrite it as

$$(a-b)\sum_{k=0}^{n} a^{n-k}b^k = a^{n+1} - b^{n+1}.$$

Base case $n = 1$: $(a-b)(a+b) = a^2 - b^2$, True.

Induction step: let $n \geqslant 1$ be arbitrary but fixed and suppose that the formula holds for n. We have

$$\begin{aligned}(a-b)\sum_{k=0}^{n+1} a^{n+1-k}b^k &= (a-b)\sum_{k=0}^{n} a^{n+1-k}b^k + (a-b)b^{n+1}\\ &= (a-b)a\sum_{k=0}^{n} a^{n-k}b^k + (a-b)b^{n+1}\\ &= a(a^{n+1} - b^{n+1}) + (a-b)b^{n+1} = a^{n+2} - b^{n+2},\end{aligned}$$

so the formula holds for $n + 1$.

Solution 1.20.

(a) We proceed by induction. Let $p(n)$ be the statement "there exists $m \in \mathbf{Z}$ such that $n^3 - n + 3 = 3m$".

Base case $n = 0$: the statement $p(0)$ is True as $0^3 - 0 + 3 = 3 = 3(1)$.

Induction step: let $k \geqslant 0$ be arbitrary but fixed and suppose $p(k)$ is True. That is, assume that there exists $m \in \mathbf{Z}$ such that $k^3 - k + 3 = 3m$.

We have

$$\begin{aligned}(k+1)^3 - (k+1) + 3 &= k^3 + 3k^2 + 2k + 3\\ &= (k^3 - k + 3) + 3(k^2 + 3k)\\ &= 3m + 3(k^2 + k) && (p(k) \text{ is True})\\ &= 3(m + k^2 + k).\end{aligned}$$

Therefore 3 divides $(k+1)^3 - (k+1) + 3$. Therefore $p(k+1)$ is True.

The other parts of this question are similar, except (d). Recall that when we talk about factors of polynomials, we mean something slightly different to factors of integers. It may help to know that

$$x^n - y^n = x^n - x^{n-1}y + x^{n-1}y - y^n = x^{n-1}(x-y) + y(x^{n-1} - y^{n-1}).$$

Solution 1.21.

(a) In summation notation the claim is:

$$\sum_{j=1}^{n-1} j^3 < \frac{n^4}{4} < \sum_{j=1}^{n} j^3 \qquad \text{for all } n \geqslant 2.$$

We proceed by induction on n.

Base case $n = 2$: the claim is $1^3 < 2^4/4 < 1^3 + 2^3$, in other words $1 < 4 < 9$, which is clearly True.

Induction step: let $n \geqslant 2$ be arbitrary but fixed and suppose that the claim holds for n. Then

$$\begin{aligned}\sum_{j=1}^{(n+1)-1} j^3 = \left(\sum_{j=1}^{n-1} j^3\right) + n^3 < \frac{n^4}{4} + n^3 = \frac{n^4 + 4n^3}{4} \\ < \frac{n^4 + 4n^3 + 6n^2 + 4n + 1}{4} = \frac{(n+1)^4}{4},\end{aligned}$$

which is the first inequality in the claim for $(n + 1)$.

For the second inequality in the claim, we have

$$\begin{aligned}\sum_{j=1}^{n+1} j^3 = \left(\sum_{j=1}^{n} j^3\right) + (n+1)^3 > \frac{n^4}{4} + (n+1)^3 = \frac{n^4 + 4n^3 + 12n^2 + 12n + 4}{4} \\ > \frac{n^4 + 4n^3 + 6n^2 + 4n + 1}{4} = \frac{(n+1)^4}{4}.\end{aligned}$$

(b) In summation notation the claim is:

$$\sum_{j=1}^{n} \frac{1}{\sqrt{j}} \geqslant \sqrt{n} \qquad \text{for all } n \geqslant 1.$$

We proceed by induction on n.

Base case $n = 1$: the claim is $1 \geqslant 1$, True.

Induction step: let $n \geqslant 1$ be arbitrary but fixed and suppose that the claim holds for n.

We have

$$\sum_{j=1}^{n+1} \frac{1}{\sqrt{j}} = \left(\sum_{j=1}^{n}\right) + \frac{1}{\sqrt{n+1}} \geqslant \sqrt{n} + \frac{1}{\sqrt{n+1}}.$$

I claim that for all $n \in \mathbf{N}$ we have

$$\sqrt{n} + \frac{1}{\sqrt{n+1}} \geqslant \sqrt{n+1}.$$

Note that the induction step above will be complete once we have proved the claim.

To prove the claim, fix $n \in \mathbf{N}$. Then $n \geqslant 0$, so $n^2 + n \geqslant n^2$. Since both $n^2 + n$ and n^2 are non-negative, and the square-root function is monotone increasing on $[0, \infty)$, we can take square roots in the last inequality and get $\sqrt{n^2+n} \geqslant \sqrt{n^2} = n$. Adding 1 on both sides:

$$\sqrt{n(n+1)} + 1 \geqslant n + 1.$$

Dividing by the positive number $\sqrt{n+1}$ on both sides:

$$\sqrt{n} + \frac{1}{\sqrt{n+1}} \geqslant \sqrt{n+1},$$

which is the claim we wanted to prove.

Solution 1.22.

(a) I claim that $n_0 = 0$.

We proceed by induction.

Base case $n = 0$: the inequality is $1 \leqslant 1$, True.

Induction step: let $k \geqslant 0$ be arbitrary but fixed and suppose that $1 + 2k \leqslant 3^k$.

Then

$$3^{k+1} = 3 \cdot 3^k \geqslant 3(1 + 2k) = 3 + 6k \geqslant 3 + 2k = 1 + 2(k + 1),$$

so the inequality holds for $k + 1$. (In the above algebraic manipulations we used the fact that $k \geqslant 0$.)

(b) I claim that $n_0 = 4$.

We proceed by induction.

Base case $n = 4$: the inequality is $24 \geqslant 16$, True.

Induction step: let $k \geqslant 4$ be arbitrary but fixed and suppose that $k! > 2^k$.

Then

$$(k + 1)! = (k + 1) \cdot k! \geqslant (k + 1)2^k \geqslant (1 + 1)2^k = 2^{k+1},$$

so the inequality holds for $k + 1$. (We used the fact that $k \geqslant 4$.)

(c) I claim that no such n_0 exists.

In fact, we will proceed by induction to prove that the opposite inequality

$$(2n - 1)^2 > n(n + 1)$$

is True for all $n \geqslant 2$.

Base case $n = 2$: the inequality is $9 > 6$, True.

Induction step: let $k \geqslant 2$ be arbitrary but fixed and suppose that

$$(2k-1)^2 > k(k+1).$$

Then

$$\begin{aligned}\big(2(k+1)-1\big)^2 &= (2k+1)^2 = 4k^2+4k+1 = (4k^2-4k+1)+8k = (2k-1)^2+8k \\ &> k(k+1)+8k = k^2+9k = k^2+3k+6k > k^2+3k+2 \\ &= (k+1)(k+2),\end{aligned}$$

where we used the fact that $k \geqslant 2$ to deduce that $6k > 2$.

We conclude that the inequality holds for $k+1$.

(d) I claim that $n_0 = 6$.

We proceed by induction.

Base case $n = 6$: the inequality is $720 > 432$, True.

Induction step: let $k \geqslant 6$ be arbitrary but fixed and suppose that $k! > 2k^3$.

We have

$$\begin{aligned}(k+1)! &= (k+1)k! > (k+1)2k^3 = 2k^4+2k^3 = 2k^3+k^4+\frac{1}{2}k^4+\frac{1}{2}k^4 \\ &> 2k^3+6k^2+6k+2 = 2(k+1)^3,\end{aligned}$$

where we used the fact that $k \geqslant 6$ to deduce that

$$\begin{aligned}k^2 \geqslant 36 &\Rightarrow k^4 \geqslant 36k^2 > 6k^2 \\ k^3 \geqslant 216 &\Rightarrow \frac{1}{2}k^4 \geqslant 108k > 6k \\ k^4 \geqslant 1296 &\Rightarrow \frac{1}{2}k^4 > 2.\end{aligned}$$

We are done, since the inequality holds for $k+1$.

Solution 1.23. We proceed by induction.

Base case $n = 0$: $\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}^0 = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$, True.

Induction step: let $n \in \mathbf{N}$ be arbitrary but fixed, and suppose the equality holds for n. Then

$$\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}^{n+1} = \begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}^{n}\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\ n & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix} = \begin{bmatrix}1 & 0\\ n+1 & 1\end{bmatrix}.$$