3 Sequences

Convergence

Exercise 3.1. Use Definition 3.6 to prove the following:

(a) $\dfrac{1}{n+2} \longrightarrow 0$ (b) $\dfrac{1}{(n+4)^2} \longrightarrow 0$ (c) $\dfrac{n^2}{2n^2+4} \longrightarrow \dfrac{1}{2}$.

Exercise 3.2. For the following sequences, guess the limit, then verify the result using Definition 3.6.

(a) $\dfrac{1}{n^2+1}$

(b) $\dfrac{2n}{n+1}$

(c) $\dfrac{3n}{2n+5}$

(d) $\dfrac{n^2}{2n^2+3}$.

Exercise 3.3. Without appealing to the Algebra of Limits Theorem 3.15, prove the following:

$$\text{If } u_n \longrightarrow L, \text{ then } 2u_n \longrightarrow 2L.$$

Exercise 3.4. Prove the difference part of the Algebra of Limits Theorem 3.15:

Let (x_n) and (y_n) be convergent sequences with $x_n \longrightarrow \alpha$ and $y_n \longrightarrow \beta$, where $\alpha, \beta \in \mathbf{R}$. Then $x_n - y_n \longrightarrow \alpha - \beta$.

[**Hint**: Modify the proof of the sum part of the Algebra of Limits.]

Exercise 3.5. Prove the remaining case of the product formula from the Algebra of Limits Theorem 3.15:

Suppose (x_n) and (y_n) are convergent sequences with $x_n \longrightarrow 0$ and $y_n \longrightarrow \beta$, where $\beta \in \mathbf{R}$. Then $x_n y_n \longrightarrow 0$.

[**Hint**: Follow the proof in Theorem 3.15 and modify it accordingly.]

Exercise 3.6. Let (x_n) be a convergent sequence with $x_n \longrightarrow \alpha$, where $\alpha \in \mathbf{R}$. Suppose $\alpha \neq 0$ and $x_n \neq 0$ for all $n \in \mathbf{N}$. Without using the Algebra of Limits Theorem 3.15, prove that $1/x_n \longrightarrow 1/\alpha$.

Use this result and the product part of the Algebra of Limits to deduce the quotient part of the Algebra of Limits.

Exercise 3.7. Prove the Sandwich Theorem, which is the following statement:

> Let (a_n), (b_n), and (c_n) be sequences. Suppose that $a_n \longrightarrow L$, $c_n \longrightarrow L$, and $a_n \leqslant b_n \leqslant c_n$ for all $n \in \mathbf{N}$. Then $b_n \longrightarrow L$.

Subsequences

Exercise 3.8. Prove that the following sequences diverge.

(a) $e_n = \dfrac{n+1}{\sqrt{n}}$

(b) $g_n = 1 + (-1)^n$.

Exercise 3.9. For each of the following sequences, find all r such that a subsequence converges to r. Draw a conclusion about the convergence or otherwise of the sequence.

(a) $a_n = (-1)^n \dfrac{n}{n+1}$ (b) $b_n = \sin\left(\dfrac{n\pi}{2}\right) \cdot \dfrac{n}{n+1}$ (c) $c_n = (-1)^{n+1} \dfrac{n^2}{n^3+5}$.

Exercise 3.10. Let (a_n) be a sequence such that the subsequences (a_{2n}) of even terms and (a_{2n+1}) of odd terms converge to the same limit L. Show that (a_n) converges to L.

Exercise 3.11. Prove Lemma 3.20:

Let (x_n) be a sequence and $\left(x_{n_k}\right)$ a subsequence of (x_n). Then $n_k \geqslant k$ for all $k \in \mathbf{N}$.

Exercise 3.12. Let (x_n) be a sequence. A *tail* of (x_n) is a subsequence of the form $(x_{k+n_0})_k$ for some $n_0 \in \mathbf{N}$.

Let $L \in \mathbf{R}$.

(a) Prove that if (x_n) converges to L, then every tail of (x_n) converges to L.

(b) Prove that if there exists a tail of (x_n) that converges to L, then (x_n) itself converges to L.

Exercise 3.13. Let (x_n) be a sequence. A *modification* of (x_n) is a sequence (y_n) obtained as follows: fix an injective function $\varphi : \mathbf{N} \longrightarrow \mathbf{N}$, and let $y_n = x_{\varphi(n)}$ for all $n \in \mathbf{N}$.

Prove that if (x_n) converges to $L \in \mathbf{R}$, then so does any modification (y_n).

[**Hint**: Start by showing that for injective function $\varphi : \mathbf{N} \longrightarrow \mathbf{N}$ and any $M \in \mathbf{N}$, the inverse image $\varphi^{-1}\big(\{0, 1, \dots, M\}\big)$ is a finite set.]

Does the converse hold?

Monotone sequences

Exercise 3.14. Classify each sequence as monotone increasing, monotone decreasing, neither, or both.

(a) $(\sqrt{n})$

(b) $\left(\frac{(-1)^n}{n}\right)$

(c) $(n + \frac{1}{n})$

(d) $(3, 3, 3, 3, \dots)$

(e) $(\frac{1}{n})$.

From the monotone ones, which are convergent?

Exercise 3.15. Let (a_n) be a sequence that is both monotone increasing and decreasing. What can you say about the convergence of the sequence?

Cauchy sequences

Exercise 3.16. Show that the following sequences are Cauchy:

(a) $(f_n) = (4 + \frac{(-1)^n}{3n})$;

(b) $(f_n) = (\dfrac{n+3}{2n-1})$.

Exercise 3.17. Consider the sequence (f_n) defined recursively by $f_1 = 1$ and, for all $n \geqslant 1$, $f_{n+1} = 1/(1 + f_n)$.

(a) Using induction, prove that $1/2 \leqslant f_n \leqslant 1$ for all $n \geqslant 1$.

(b) Show that $|f_{n+2} - f_{n+1}| \leqslant \frac{4}{9} \cdot |f_{n+1} - f_n|$ for all $n \geqslant 1$.

(c) Using the previous part and results from the lectures, show that (f_n) converges. Compute the limit.

Exercise 3.18. Consider the cubic equation $x^3 - 6x + 3 = 0$. Find a contractive sequence that converges to a solution $L \in (0, 1)$ of this equation.

Exercise 3.19. For each item below, give an example or show it does not exist:

(a) A bounded sequence with an unbounded subsequence.

(b) An unbounded sequence with a bounded subsequence.

(c) A monotone sequence that is not Cauchy.

(d) A bounded sequence that not Cauchy.

(e) A bounded monotone sequence that is not Cauchy.

(f) A Cauchy sequence that is not bounded.

(g) A Cauchy sequence that is not monotone.

(h) A Cauchy sequence that has an unbounded subsequence.

(i) An unbounded sequence with a Cauchy subsequence.

(j) A bounded sequence with a divergent monotone subsequence.

(k) A divergent sequence with both an monotone increasing subsequence and a monotone decreasing subsequence.

(l) A convergent sequence (f_n) for which $f_n \not\longrightarrow \mathsf{sup}\{f_n : \ n \in \mathrm{N}^+\}$ and $f_n \not\longrightarrow \mathsf{inf}\{f_n : \ n \in \mathrm{N}^+\}$.

(m) A convergent monotone sequence (f_n) for which $f_n \not\longrightarrow \mathsf{sup}\{f_n : \ n \in \mathrm{N}^+\}$ and $f_n \not\longrightarrow \mathsf{inf}\{f_n : \ n \in \mathrm{N}^+\}$.

Limit points of sets

If $E \subseteq \mathbf{R}$, we say that $a \in \mathbf{R}$ is a *limit point* of E if for every $\delta > 0$, there exists $x \in E$ such that $x \neq a$ and $|x - a| < \delta$.

(Note: a limit point a of E need not be an element of E.)

Exercise 3.20. Find all the limit points of the following subsets of $\mathbf{R}$:

(a) $E = \left\{ \frac{1}{n} : \ n \in \mathbf{N} \right\}$

(b) $E = \left\{ \frac{1}{m} + \frac{1}{n} : \ m, n \in \mathbf{N} \right\}$

(c) $E = \left\{ (-1)^n \left(1 + \frac{1}{n} \right) : \ n \in \mathbf{N} \right\}$

(d) $E = \mathbf{Z}$

(e) $E = \mathbf{Q}$.

Exercise 3.21. We aim to prove the following statement:

"Let $E \subseteq \mathbf{R}$ and let $a \in \mathbf{R}$. Then a is a limit point of E if and only if there exists a sequence (x_n) such that $x_n \in E$ and $x_n \neq a$ for all $n \in \mathbf{N}$ and $x_n \longrightarrow a$ as $n \longrightarrow \infty$."

(a) First we show the "if" part. Suppose that there exists a sequence (x_n) such that $x_n \in E$ and $x_n \neq a$ for all $n \in \mathbf{N}$ and $x_n \longrightarrow a$ as $n \longrightarrow \infty$. Show that a is a limit point of E.

(b) Next we show the "only if" part. Suppose that a is a limit point of E.

  i. Use the definition of limit point to explain why for each $n \in \mathbf{N}$ there exists $x_n \in E$ such that $x_n \neq a$ and $|x_n - a| < 1/n$.

  ii. Use Part i. to show that $x_n \longrightarrow a$.

Answers

Solution 3.1. Hints:

(a) Choose $M > \left\lceil \frac{1}{\varepsilon} \right\rceil - 2.$

(b) Choose $M > \left\lceil \frac{1}{\sqrt{\varepsilon}} \right\rceil - 4.$

(c) Choose $M > \left\lceil \frac{\sqrt{(2/\varepsilon)-4}}{2} \right\rceil.$

These answers are not unique. We do not need to choose the smallest possible value of M. For example, in (a) one could choose $M > \left\lceil \frac{1}{\varepsilon} \right\rceil$ as $\left\lceil \frac{1}{\varepsilon} \right\rceil > \left\lceil \frac{1}{\varepsilon} \right\rceil - 2.$

Solution 3.2. Limits:

(a) 0 (b) 2 (c) $\frac{3}{2}$ (d) $\frac{1}{2}$.

Solution 3.3. Proceed as in the proof of part i of the Algebra of Limits Theorem. Notice $2u_n = u_n + u_n$.

Solution 3.4. Let $\varepsilon > 0$. Then there exists $M_x \in \mathbf{N}$ such that for all $n \in \mathbf{N}$ we have

$$n > M_x \Rightarrow |x_n - \alpha| < \frac{\varepsilon}{2}.$$

Similarly, there exists $M_y \in \mathbf{N}$ such that for all $n \in \mathbf{N}$ we have

$$n > M_y \Rightarrow |y_n - \beta| < \frac{\varepsilon}{2}.$$

Let $M = \max\{M_x, M_y\}$ and $n \in \mathbf{N}$. Then

$$\begin{aligned} n > M \Rightarrow |(x_n - y_n) - (\alpha - \beta)| &= |(x_n - \alpha) - (y_n - \beta)| \\ &\leqslant |(x_n - \alpha)| + |y_n - \beta| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Solution 3.5. The sequence (y_n) is convergent, therefore bounded by some $C > 0$. Let $\varepsilon > 0$. As $(x_n) \longrightarrow 0$, there exists some $M \in \mathbf{N}$ such that for all $n \in \mathbf{N}$,

$$n > M \Rightarrow |x_n| < \frac{\varepsilon}{C}.$$

But then,

$$|x_n y_n| = |x_n||y_n| \leqslant |x_n|C < \frac{\varepsilon}{C} \cdot C = \varepsilon$$

for all $n > M$. So $(x_n y_n) \longrightarrow 0$.

Solution 3.6. First, we bound (x_n) away from 0. Since $\alpha \neq 0$, $\dfrac{1}{2}|\alpha| > 0$. As $(x_n) \longrightarrow \alpha$, we can pick some M_1 such that for all $n \in \mathbf{N}$,

$$n > M_1 \Rightarrow |x_n - \alpha| < \frac{1}{2}|\alpha|.$$

Using the triangle inequality, one sees that

$$|\alpha| = |\alpha - x_n + x_n| \leqslant |x_n - \alpha| + |x_n| < \frac{1}{2}|\alpha| + |x_n|.$$

So for $n > M_1$, we have that $|x_n| > \dfrac{1}{2}|\alpha|$.

Let $\varepsilon > 0$. Choose another $\varepsilon' = \mathsf{min}\left\{\varepsilon, \dfrac{1}{2}|\alpha|^2\varepsilon\right\}$. Then $\varepsilon' > 0$ and there exists an $M_2 \in \mathbf{N}$ such that for all $n \in \mathbf{N}$,

$$n > M_2 \Rightarrow |x_n - \alpha| < \varepsilon'.$$

Choose $M = \mathsf{max}\{M_1, M_2\}$. Let $n \in \mathbf{N}$. If $n > M$, then

$$\begin{aligned} \left|\frac{1}{x_n} - \frac{1}{\alpha}\right| &= \left|\frac{\alpha - x_n}{\alpha x_n}\right| = \frac{|\alpha - x_n|}{|\alpha||x_n|} \\ &< \frac{\varepsilon'}{|\alpha| \cdot \frac{1}{2}|\alpha|} = \frac{\varepsilon'}{\frac{1}{2}|\alpha|^2} \\ &\leqslant \varepsilon. \end{aligned}$$

So indeed $1/(x_n) \longrightarrow 1/\alpha$. Let $(y_n) \longrightarrow \beta$ and using the product part of Algebra of limits, we get that

$$(y_n)/(x_n) = (y_n)(1/x_n) \longrightarrow \beta(1/\alpha) = \beta/\alpha$$

as required.

Solution 3.7. Note that since $a_n \leqslant b_n \leqslant c_n$ for all $n \in \mathbf{N}$, we have

$$a_n - L \leqslant b_n - L \leqslant c_n - L \qquad \text{for all } n \in \mathbf{N}.$$

Let $\varepsilon > 0$.

Since $a_n \longrightarrow L$, there exists $M_a \in \mathbf{N}$ such that for all $n \in \mathbf{N}$ we have

$$n > M_a \Rightarrow |a_n - L| < \varepsilon \Rightarrow -\varepsilon < a_n - L < \varepsilon.$$

Since $c_n \longrightarrow L$, there exists $M_c \in \mathbf{N}$ such that for all $n \in \mathbf{N}$ we have

$$n > M_c \Rightarrow |c_n - L| < \varepsilon \Rightarrow -\varepsilon < c_n - L < \varepsilon.$$

Let $M = \max\{M_a, M_c\}$. Let $n \in \mathbf{N}$ be such that $n > M$. Then

$$-\varepsilon < a_n - L \leqslant b_n - L \leqslant c_n - L < \varepsilon,$$

therefore $|b_n - L| < \varepsilon$.

We conclude that $b_n \longrightarrow L$.

Solution 3.8. (a) The sequence is unbounded. For any $r \in \mathbf{R}$ we can find $n \in \mathbf{Z}_{\geqslant 1}$ so that $f_n > r$. (For instance, choose n so that $n > r^2$.)

(b) There are two subsequences that converge to different limits.

Solution 3.9.

(a) $-1,\ 1;$ divergent.

(b) $-1,\ 0,\ 1;$ divergent.

(c) $0;$ convergent.

Solution 3.10. Let $\varepsilon > 0$.

Since $a_{2n} \longrightarrow L$ there exists $M_e \in \mathbf{N}$ such that if $n > M_e$ then $|a_{2n} - L| < \varepsilon$.

Since $a_{2n+1} \longrightarrow L$ there exists $M_o \in \mathbf{N}$ such that if $n > M_o$ then $|a_{2n+1} - L| < \varepsilon$.

Let $M = \max\{2M_e, 2M_o + 1\}$. Suppose $m \in \mathbf{N}$ with $m > M$. There are two cases:

- m is even: $m = 2n$ for some $n \in \mathbf{N}$. Since $m > M \geqslant 2M_e$, we have $n > M_e$, so

$$|a_m - L| = |a_{2n} - L| < \varepsilon.$$

- m is odd: $m = 2n + 1$ for some $n \in \mathbf{N}$. Since $m > M \geqslant 2M_o$, we have $n > M_o$, so

$$|a_m - L| = |a_{2n+1} - L| < \varepsilon.$$

In both cases we concluded that if $m > M$ then $|a_m - L| < \varepsilon$, so $a_m \longrightarrow L$.

Solution 3.11. We proceed by induction on $k \in \mathbf{N}$.

Base case $k = 0$: $n_0 \geqslant 0$ is True, since $n_0 \in \mathbf{N}$.

Induction step: let $j \in \mathbf{N}$ be arbitrary but fixed and suppose $n_j \geqslant j$. By the definition of a subsequence, we have $n_{j+1} > n_j$, so $n_{j+1} > j$. Since $n_{j+1} \in \mathbf{N}$, this implies that $n_{j+1} \geqslant j + 1$.

Solution 3.12.

(a) Let $\varepsilon > 0$. Then there exists an $M_1 \in \mathbf{N}$ such that for all $n \in \mathbf{N}$,

$$n > M_1 \Rightarrow |x_n - L| < \varepsilon.$$

Let $(x_{k+n_0})_k$ be some tail with $n_0 \in \mathbf{N}$. Choose $M = \max\{n_0, M_1\}$, then for all $n > M$. We have $x_n \in (x_{k+n_0})_k$ (so the term x_n is actually in the tail), and

$$|x_n - L| < \varepsilon.$$

So indeed every tail converges to L.

(b) Let $n_0 \in \mathbf{N}$ be such that $(x_{k+n_0})_k$ is a tail that converges to L. Let $\varepsilon > 0$. Then there exists an M_1 such that for all $n \in \mathbf{N}$,

$$n > M_1 \Rightarrow |x_{n+n_0} - L| < \varepsilon.$$

Choose $M = M_1 + n_0$. Then for all $n > M$, $|x_n - L| < \varepsilon$. So if some tail converges to L, then the whole sequence converges to L.

Solution 3.13. We start by proving the statement in the hint. We have

$$\begin{aligned}\varphi^{-1}\big(\{0, 1, \dots, M\}\big) &= \big\{n \in \mathbf{N} : \ \varphi(n) \in \{0, 1, \dots, M\}\big\} \\ &= \{n \in \mathbf{N} : \ \varphi(n) = y, 0 \leqslant y \leqslant M\} \\ &= \{n \in \mathbf{N} : \ \varphi(n) = 0\} \cup \dots \cup \{n \in \mathbf{N} : \ \varphi(n) = M\}.\end{aligned}$$

For each y with $0 \leqslant y \leqslant M$, the set $\{n \in \mathbf{N} : \ \varphi(n) = y\}$ has at most one element (by the injectivity of φ). Therefore the set we are interested in is the union of finitely many sets with 0 or 1 elements, hence is finite.

We now prove the statement in the question.

Let $x_n \longrightarrow L$ and let (y_n) be a modification of (x_n), given by some injective function $\varphi : \ \mathbf{N} \longrightarrow \mathbf{N}$.

Let $\varepsilon > 0$.

Since $x_n \longrightarrow L$, there exists $M \in \mathbf{N}$ such that for all $n > M$ we have

$$|x_n - L| < \varepsilon.$$

By the statement in the hint, the inverse image $\varphi^{-1}\big(\{0, \dots, M\}\big)$ is a finite set, so it has a maximal element N. (If the set is empty, just take $N = 0$.) For all $n > N$, we have $\varphi(n) > M$, so

$$|y_n - L| = |x_{\varphi(n)} - L| < \varepsilon.$$

The converse certainly does not hold. For instance, take $(x_n) = (1, 0, 1, 0, 1, 0, \dots)$ and $\varphi(n) = 2n$, then the sequence $(y_n) = (0, 0, 0, \dots)$ converges to 0 but (x_n) does not converge.

Solution 3.14.

(a) monotone increasing

(b) neither

(c) monotone increasing

(d) both

(e) monotone decreasing.

(a) and (c) are not bounded, hence divergent. (d) and (e) are bounded (and monotone), hence convergent.

Solution 3.15. If $a_n \leqslant a_{n+1}$ and $a_n \geqslant a_{n+1}$ for all $n \in \mathbf{N}$, then $a_n = a_{n+1}$ for all $n \in \mathbf{N}$. Therefore every term of (a_n) is equal to a_1, the sequence is constant, and it converges to a_1.

Solution 3.16.

(a) Choose M so that $\dfrac{1}{M} < \dfrac{3\varepsilon}{2}$. Then for all $n, m > M$ one has:

$$\begin{aligned}
|f_n - f_m| &= \left|\left(4 + \frac{(-1)^n}{3n}\right) - \left(4 + \frac{(-1)^m}{3m}\right)\right| = \left|\frac{(-1)^n}{3n} - \frac{(-1)^m}{3m}\right| \\
&\leqslant \left|\frac{(-1)^n}{3n}\right| + \left|\frac{(-1)^m}{3m}\right| = \frac{1}{3n} + \frac{1}{3m} \\
&\leqslant \frac{1}{3M} + \frac{1}{3M} = \frac{2}{3M} < \varepsilon
\end{aligned}$$

(b) Pick $M \geqslant \left\lceil \dfrac{14}{\varepsilon} \right\rceil$. Then for all $n, m > M$ one has:

$$\begin{aligned}
|f_n - f_m| &= \left|\frac{n+3}{2n-1} - \frac{m+3}{2m-1}\right| = \left|\frac{7m - 7n}{(2m-1)(2n-1)}\right| \\
&\leqslant \left|\frac{7m - 7n}{mn}\right| \leqslant \frac{7m}{mn} + \frac{7n}{nm} \\
&\leqslant \frac{7}{n} + \frac{7}{m} < \frac{7}{M} + \frac{7}{M} \\
&< \varepsilon
\end{aligned}$$

The inequality on the second line comes from $2m - 1 \geqslant m$ for $m \geqslant 1$, which is guaranteed by $m > M \geqslant 0$.

Solution 3.17.

(a) For the inductive step, note that $1/2 \leqslant x \leqslant 1$ implies $1 < 1 + x \leqslant 2$.

(b) From the previous part, conclude that $1/2 \leqslant 1/(1 + f_n) \leqslant 2/3$ for all n. A bit of algebra shows that $|f_{n+2} - f_{n+1}| = \frac{|f_{n+1}-f_n|}{(1+f_{n+1})(1+f_n)}$. Combine these two to obtain the bound.

(c) The sequence is contractive, and therefore convergent. Let f be the limit. Applying the Algebra of Limits Theorem, we see that $f = 1/(1 + f)$, or $f^2 + f - 1 = 0$. This quadratic equation has one positive root $f = (\sqrt{5} - 1)/2$. (This is the reciprocal of the Golden Ratio.)

Solution 3.18. Choose some x_1 with $0 < x_1 < 1$, and inductively define a sequence by $x_{n+1} = \frac{1}{6}(x_n^3 + 3)$. Proceeding as in the example given in lectures, you can show that this sequence is contractive with $c = 1/2$, and $0 < x_n < 1$ for all n. If L is its limit, then $0 \leqslant L \leqslant 1$ and the Algebra of Limits Theorem shows that $L = \frac{1}{6}(L^3 + 3)$, and thus $L^3 - 6L + 3 = 0$.

Solution 3.19.

(a) If (f_n) is bounded then there exists $C \in \mathbf{R}$ so that $|f_n| \leqslant C$ for all $n \in \mathrm{N}^+$. Therefore the terms of any subsequence (f_{n_k}) must also be bounded by C. Thus no example exists.

(b) $(1, 0, 2, 0, 3, 0, 4, \dots)$

(c) $(1, 2, 3, \dots)$

(d) $(-1, 1, -1, \dots)$

(e) Does not exist. A bounded monotone sequence must converge. And so must be Cauchy

(f) Does not exist. A Cauchy sequence must converge. All convergent sequences are bounded.

(g) $\left(\frac{(-1)^n}{n}\right)$

(h) Does not exist. A Cauchy sequence must converge. All subsequences of a convergent sequence must converge. All convergent sequences are bounded.

(i) $(1, 0, 2, 0, 3, 0, 4, \dots)$

(j) A divergent monotone subsequence is not bounded. See (a)

(k) $(-1, 1, -1, \dots)$

(l) $(100, -100, 0, 0, \dots)$

(m) Does not exist. In the proof that a bounded increasing monotone converges, one shows that it converges to $\mathsf{sup}\,\{f_n :\ n \in \mathrm{N}^+\}$

Solution 3.20. (a) 0

(b) $\left\{\frac{1}{n} :\ n \in \mathbf{N}\right\} \cup \{0\}$

(c) $1, -1$

(d) None

(e) $\mathbf{R}$.

Solution 3.21.

(a) Let $E \subseteq \mathbf{R}$ and let $a \in \mathbf{R}$. Let (x_n) be a sequence such that $x_n \in E$ and $x_n \neq a$ for all $n \in \mathbf{N}$ and $x_n \longrightarrow a$ as $n \longrightarrow \infty$.

Let $\varepsilon > 0$. Since $x_n \longrightarrow a$

$$\text{there exists } M \in \mathbf{N} \text{ such that if } n \in \mathbf{N} \text{ with } n > M \text{ then } |x_n - a| < \varepsilon.$$

Therefore, $x_n \in E$, $x_n \neq a$, and $|x_n - a| < \varepsilon$. That is, there exists a point $x \in E$ (namely $x = x_n$) such that $x \neq a$ and $|x - a| < \varepsilon$. Therefore, a is a limit point of E.

(b) Suppose that a is a limit point of E.

i. Since a is a limit point of E

$$\text{for every } \delta > 0 \text{ there exists } x \in E \text{ such that } x \neq a \text{ and } |x - a| < \delta.$$

Let $\delta = 1/n$. Then there exists $x_n \in E$ such that $x_n \neq a$ and $|x_n - a| < 1/n$.

ii. Hint: Use the Archimedean Principle.