2 Set theory

Sets

Exercise 2.1. Let $A = \{1, 2, 3, 4\}$, $B = \{1, 3, 5, 7\}$ and $C = \{2, 3\}$.

(a) Find the following sets:

i. $A \cup B$
ii. $A \cap B$
iii. $A \cup C$
iv. $A \cap C$
v. $A \cup B \cup C$
vi. $A \cap B \cap C$.

(b) Find the following sets:

i. $B \times C$
ii. $A \times \varnothing$.

(c) Which of the following statements are True?

i. $3 \subseteq B$
ii. $\varnothing \subseteq A$
iii. $7 \in B \cup C$
iv. $C \subseteq B$
v. $C = B \cup C$
vi. $C \in A$
vii. $(\forall a \in A)\ a \leqslant 4$
viii. $(\forall b \in B)[(\exists c \in C)\ b - c \geqslant 0]$.

Exercise 2.2. Show that set union is an *associative* operation, that is for any three sets A, B, C we have

$$\big(A \cup B\big) \cup C = A \cup \big(B \cup C\big).$$

Exercise 2.3. Let A, B and C be sets. Let $\mathcal{U} = A \cup B \cup C$. Prove the following statements.

(Please write your proofs using the definitions of $=$, $\subseteq$, $\cup$, $\cap$, etc. from the lectures on set theory. Do not use set algebra.)

(a) $A \subseteq [(A \cap B) \cup (A \cap (\mathcal{U} \setminus B))]$

(b) $A \subseteq B \Rightarrow (C \cap A) \subseteq (C \cap B)$

(c) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(d) $A \subseteq B$ if and only if $A \cup B \subseteq B$.

Exercise 2.4. We say that two sets S_1 and S_2 are *disjoint* if $S_1 \cap S_2 = \varnothing$.

Show that given any two sets A and B, the sets

$$S_1 := A \setminus B, \qquad S_2 := A \cap B, \qquad S_3 := B \setminus A$$

are pairwise disjoint.

What is their union $S_1 \cup S_2 \cup S_3$?

Exercise 2.5 (Cantor Set)**.** We are going to construct an unusual set made famous by Georg Cantor.

We begin with the set $C_0 = [0, 1]$.

(a) Draw this set on the real number line. What is its length?

(b) Construct C_1 by removing the middle third of C_0 as an open interval. In other words, remove $(\frac{1}{3}, \frac{2}{3})$. Write C_1 as the union of two intervals. Draw C_1 on the real line, and determine its length.

(c) Construct C_2 by removing the middle third of each segment in C_1. C_2 should have 4 segments. Write C_2 as a union of 4 intervals. Draw it on the real line, and determine its length.

(d) By now you probably understand the pattern. Draw a few more: C_3, C_4, C_5, etc.

(e) The *Cantor set* C is what remains after you repeat this process **ad infinitum**. Find several points that are in the set C. Find several points that are not in the set C.

(f) The Cantor set is a simple example of a *fractal*. It exhibits self-similarity: meaning if you zoom in on sections, you see the overal shape repeating itself. Type "fractal cauliflower" into Google to see another example of a fractal.

Functions

Exercise 2.6. Let $f, g : A \longrightarrow B$ be two functions from a set A to a set B.

Prove that $f = g$ if and only if $f(x) = g(x)$ for all $x \in A$.

[**Hint**: f and g are relations, hence subsets of $A \times B$.]

Exercise 2.7. Let $f : A \longrightarrow B$ be a function from a set A to a set B.

Prove that:

(a) f is injective if and only if $\forall y \in B$, the equation $f(x) = y$ has at most one solution $x \in A$;

(b) f is surjective if and only if $\forall y \in B$, the equation $f(x) = y$ has at least one solution $x \in A$;

(c) f is bijective if and only if $\forall y \in B$, the equation $f(x) = y$ has exactly one solution $x \in A$.

Exercise 2.8. Let $f : A \longrightarrow B$ be a function from a set A to a set B.

Then f is bijective if and only if it is invertible.

Exercise 2.9. Let X be a nonempty set and let y be an element that may or may not be in X.

(a) How many functions $X \longrightarrow \varnothing$ are there?

(b) How many functions $\varnothing \longrightarrow X$ are there?

(c) How many functions $X \longrightarrow \{y\}$ are there?

(d) How many functions $\{y\} \longrightarrow X$ are there?

Exercise 2.10. Consider two functions $f : A \longrightarrow B$ and $g : B \longrightarrow C$.

(a) Suppose $g \circ f$ is injective. Does it follow that f must be injective? That g must be injective?

(b) Suppose $g \circ f$ is surjective. Does it follow that f must be surjective? That g must be surjective?

(c) Suppose $g \circ f$ is bijective. What can you conclude about f and g?

Exercise 2.11. Let $f : A \longrightarrow C$ and $g : B \longrightarrow D$ be bijective functions. Give a bijective function $h : A \times B \longrightarrow C \times D$ (and prove that it is bijective).

Exercise 2.12. Let $f : A \longrightarrow B$ be a function and let $X \subseteq A$. Recall from Definition 2.27 that the *image of X under f* is the subset of B defined by

$$f(X) = \{b \in B : \ b = f(x) \text{ for some } x \in X\}.$$

Suppose we have another subset $Y \subseteq A$.

(a) Prove that
$$f(X \cup Y) = f(X) \cup f(Y).$$

(b) Prove that
$$f(X \cap Y) \subseteq f(X) \cap f(Y).$$
Does the opposite inclusion always hold?

Exercise 2.13. Let $f : A \longrightarrow B$ be a function and let $X \subseteq B$. Recall from Definition 2.27 that the *inverse image of X under f* is the subset of A defined by

$$f^{-1}(X) = \{a \in A : \ f(a) \in X\}.$$

Suppose we have another subset $Y \subseteq B$.

(a) Prove that
$$f^{-1}(X \cup Y) = f^{-1}(X) \cup f^{-1}(Y).$$

(b) Prove that

$$f^{-1}(X \cap Y) \subseteq f^{-1}(X) \cap f^{-1}(Y).$$

Does the opposite inclusion always hold?

Exercise 2.14. Let $f: A \longrightarrow B$ be a function.

(a) Prove that

$$f^{-1}(f(X)) = X \qquad \text{for every } X \subseteq A$$

if and only if f is injective.

(b) Prove that

$$f(f^{-1}(X)) = X \qquad \text{for every } X \subseteq B$$

if and only if f is surjective.

Exercise 2.15. Let A, B be two nonempty sets. Consider the functions

$$\pi_A: A \times B \longrightarrow A \quad \text{given by } \pi_A(a,b) = a \text{ for all } (a,b) \in A \times B$$
$$\pi_B: A \times B \longrightarrow B \quad \text{given by } \pi_B(a,b) = b \text{ for all } (a,b) \in A \times B.$$

(a) Prove that π_A and π_B are surjective.

(b) Prove that given any set X and any functions $f: X \longrightarrow A$ and $g: X \longrightarrow B$, there exists a unique function $h: X \longrightarrow A \times B$ such that

$$\pi_A \circ h = f \qquad \text{and} \qquad \pi_B \circ h = g.$$

(This is called the *universal mapping property* of the Cartesian product $A \times B$.)

Supremum and infimum

Exercise 2.16. Find the supremum and infimum of each set S (where $S \subseteq \mathbf{R}$), if they exist, and if they do, state whether the supremum or infimum is an element of S. You do not need to prove your answer is correct.

(a) $S = \{x : x^2 \leqslant 9\}$

(b) $S = \{x : |x-2| < 3\}$

(c) $S = \{x : |2x+1| < 5\}$

(d) $S = \{x : |x-2| < 3 \wedge |x+1| < 1\}$

(e) $S = \{x : |x+2| \leqslant 2 \vee |x| > 1\}$

(f) $S = \{x \in \mathbf{Q} : x^2 \leqslant 7\}$.

Exercise 2.17. Let $S \subseteq \mathbf{R}$ and $c \in \mathbf{R}$. Define

$$c + S = \{c + x : x \in S\} \qquad \text{and} \qquad cS = \{cx : x \in S\}.$$

Prove:

(a) if S is bounded above then $c+S$ is bounded above and $\mathsf{sup}\,(c+S) = c + \mathsf{sup}\,S$;

(b) if S is bounded below then $c+S$ is bounded below and $\mathsf{inf}\,(c+S) = c + \mathsf{inf}\,S$;

(c) if S is bounded above and $c \geqslant 0$ then cS is bounded above and $\mathsf{sup}\,(cS) = c\,\mathsf{sup}\,S$;

(d) if S is bounded below and $c \geqslant 0$ then cS is bounded below and $\mathsf{inf}\,(cS) = c\,\mathsf{inf}\,S$;

(e) if S is bounded above and $c < 0$ then cS is bounded below and $\mathsf{inf}\,(cS) = c\,\mathsf{sup}\,S$;

(f) if S is bounded below and $c < 0$ then cS is bounded above and $\mathsf{sup}\,(cS) = c\,\mathsf{inf}\,S$.

Exercise 2.18. Use the Completeness Axiom of $\mathbf{R}$ to prove that: every non-empty subset $B \subseteq \mathbf{R}$ that is bounded below in $\mathbf{R}$ has an infimum in $\mathbf{R}$.

Inequalities

Exercise 2.19. Let $a, b, c \in \mathbf{R}$ and let $a < b$. Which of the following statements are always True? Which are sometimes True? Which are never True?

(a) $a + 1 < b + 1$

(b) $a + c < b + c$

(c) $5a < 5b$

(d) $ac < bc$

(e) $\dfrac{1}{a} < \dfrac{1}{b}$

(f) $\dfrac{1}{c+a} < \dfrac{1}{c}$

(g) $c < c + a$

(h) $-a < -b$.

Exercise 2.20. Express the solutions to the following inequalities as intervals of $\mathbf{R}$.

(a) $|1 + 2x| \leqslant 4$

(b) $|x + 2| \geqslant 5$

(c) $|x - 5| < |x + 1|$

(d) $|x - 2| < 3 \vee |x + 1| < 1$

(e) $|x - 2| < 3 \wedge |x + 1| < 1$.

Exercise 2.21. Transform each of the following inequalities into an equivalent inequality free of the modulus sign, such as $a < x < b$. Simplify as much as possible.

(a) $|x| < 3$

(b) $|x - 2| < 5$

(c) $|3 - 2x| < 1$

(d) $|1 + 2x| \leqslant 3$

(e) $|x+2| \geqslant 5$

(f) $|5-x^{-1}| < 1$

(g) $|x-5| < |x+1|$

(h) $|x^2-2| \leqslant 1$

Exercise 2.22. Rewrite each of the following inequalities in terms of intervals:

(a) $|x+3| \geqslant 1$

(b) $|x-2| < 3$

(c) $|x-2| < 3$ or $|x+1| < 1$

(d) $|x-2| < 3$ and $|x+1| < 1$

(e) $|x+2| \leqslant 2$ and $|x| > 1$

(f) $|x+2| \leqslant 2$ or $|x| > 1$.

Exercise 2.23. Use the triangle inequality or other properties of inequalities to find a bound for $|f|$ on the stated interval.

(a) $f(x) = \dfrac{2x^2+1}{x+3}, \quad |x| < 1$

(b) $f(x) = \dfrac{x^3+3x+1}{10-x^3}, \quad |x+1| < 2.$

Exercise 2.24. Prove that $\forall a, b \in \mathbf{R}$ we have $|a-b| \geqslant \big||a|-|b|\big|$.

Exercise 2.25. Let $a, b \in \mathbf{R}$. If $0 < \varepsilon < \min\{|a|, |b|\}$ show that

$$\left|\frac{a+\varepsilon}{b+\varepsilon}\right| \leqslant \frac{|a|+\varepsilon}{|b|-\varepsilon}.$$

Exercise 2.26. Let $a, b \geqslant 0$, $p > 1$ and $q = p/(p-1)$. We will prove that

$$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q}.$$

(a) Consider the cases where either $a = 0$ or $b = 0$: the inequality is trivially true for these.

(b) Treat a as a real variable and define

$$f(x) = \frac{x^p}{p} + \frac{b^q}{q} - bx, \quad x > 0.$$

Show that f has a minimum at $x = b^{\frac{1}{p-1}}$ using calculus techniques.

(c) Find the value of the function at this point, and conclude $f(x) \geqslant 0$.

(d) Finish off the proof.

Real numbers and their subsets

Exercise 2.27. Let $x, y, z \in \mathbf{R}$. Using the axioms of the order on the real numbers, prove the following:

(a) If $x < y$ and $z < 0$ then $xz > yz$. In particular, $-y < -x$.

(b) If $x > 0$ then $1/x > 0$.

(c) If $x < y$, then

$$x < \frac{x+y}{2} < y.$$

Exercise 2.28. (a) For every $x \in \mathbf{R}$ we have $-x \leqslant |x|$ and $x \leqslant |x|$.

(b) If $x, y, \varepsilon \in \mathbf{R}$ with $\varepsilon \geqslant 0$, then

$$|x - y| < \varepsilon \qquad \text{if and only if} \qquad y - \varepsilon < x < y + \varepsilon.$$

(The same statement holds if we replace $<$ by $\leqslant$ everywhere.)

Exercise 2.29. For each pair of sets, write down an equation that has at least one solution in one of the sets but no solutions in the other.

(a) $\mathbf{N}$ and $\mathbf{Z}$ (b) $\mathbf{Z}$ and $\mathbf{Q}$ (c) $\mathbf{Q}$ and $\mathbf{R}$.

Exercise 2.30. Prove the statement

$$(\forall \varepsilon > 0)\ [(\exists n \in \mathbf{N})\ 1/n < \varepsilon].$$

Exercise 2.31. The objective is to prove that there exists a positive real number α such that $\alpha^2 = 2$.

Throughout this exercise, only use facts that follow from the basic algebraic and order properties of the real numbers. Be careful to **not** assume the existence of square roots (otherwise your argument would be circular, hence invalid).

(a) Let $A = \{x \in \mathbf{R} :\ x^2 \leqslant 2\}$. Show A is bounded above in $\mathbf{R}$. That is, find a specific real number β that is an upper bound for A. Prove that β is indeed an upper bound for A in $\mathbf{R}$.

(b) Explain why the supremum α of the set A exists in $\mathbf{R}$.

(c) Suppose that $\alpha^2 < 2$.

i. Use the Archimedean Principle to show that there exists $n \in \mathbf{Z}_{\geqslant 1}$ such that $(\alpha + 1/n)^2 < 2$.

ii. Conclude that α is not an upper bound of A. (This of course contradicts Part (b)!)

(d) Suppose that $\alpha^2 > 2$.

i. Use the Archimedean Principle to show that there exists $n \in \mathbf{Z}_{\geqslant 1}$ such that $(\alpha - 1/n)^2 > 2$.

ii. Conclude that α is not the least upper bound of A. (This again contradicts Part (b)!)

(e) Putting everything together, explain why $\alpha > 0$ and $\alpha^2 = 2$.

Equivalence relations (very optional)

Exercise 2.32. An *equivalence relation* $\sim$ is a way of identifying elements of a set. More precisely, given a set A and $x, y \in A$, we will write $x \sim y$ to signify that "x is *equivalent* to y", and we require this to satisfy three properties:

- $x \sim x$ for all $x \in A$ (*reflexivity*);
- if $x \sim y$ then $y \sim x$ (*symmetry*);
- if $x \sim y$ and $y \sim z$ then $x \sim z$ (*transitivity*).

(a) Let A, B be sets and $f\colon A \longrightarrow B$ a function. For $x, y \in A$, define $x \sim y$ if $f(x) = f(y)$. Show that this satisfies the properties of an equivalence relation on A.

(b) Fix a natural number n. For $k, m \in \mathbf{Z}$, define $k \sim m$ if $m - k$ is divisible by n. Show that this satisfies the properties of an equivalence relation on $\mathbf{Z}$.

(c) Fix a set Ω and let A denote the set of all subsets of Ω. For $X, Y \in A$ define $X \sim Y$ if there exists a bijective function $X \longrightarrow Y$. Show that this satisfies the properties of an equivalence relation on A.

(d) Suppose we are given an equivalence relation on a set A. For any element $x \in A$, we define the *equivalence class* of x as:

$$[x] = \{y \in A \mid x \sim y\}.$$

Show that, for any elements $x, z \in A$, their equivalence classes are either identical or disjoint, in other words:

$$\text{either } [x] = [z] \quad \text{or } [x] \cap [z] = \varnothing.$$

(e) How many distinct equivalence classes are there for the equivalence relation on **Z** defined in part (b)? Make sure to prove that the number you give (and which will probably depend on n) is the **exact** number of classes (not just an upper bound, for instance).

(f) Suppose we are given an equivalence relation on a set A, and consider the set B of equivalence classes[1]:

$$B = \{[x] \mid x \in A\}.$$

There is an obvious surjective function $\pi\colon A \longrightarrow B$ defined by $[x]$. Under what circumstances (if any) is π a bijection?

(g) Let $A = \mathbf{N} \times \mathbf{N}$ and define $(a, b) \sim (c, d)$ if $a + d = b + c$.

i. Show that this satisfies the conditions of an equivalence relation on A.

ii. Construct a bijective function $B \longrightarrow \mathbf{Z}$, where B is the set of equivalence classes. (Don't forget to prove that your function is well-defined, and that it is bijective.)

[1]A frequently used notation for the set of equivalence classes is $A/\sim$, read as "A mod tilde", and it is referred to as the *quotient* of A by the relation $\sim$

Answers

Solution 2.1.

(a)
- i. $\{1,2,3,4,5,7\}$
- ii. $\{1,3\}$
- iii. $\{1,2,3,4\} = A$
- iv. $\{2,3\} = C$
- v. $\{1,2,3,4,5,7\} = A \cup B$
- vi. $\{3\}$.

(b)
- i. $\{(1,2),(1,3),(3,2),(3,3),(5,2),(5,3),(7,2),(7,3)\}$
- ii. $\varnothing$.

(c)
- i. False: 3 is not a set, so it cannot be a subset of anything. However, we could say 3 is an element of B ($3 \in B$).
- ii. True: the empty set is a subset of all sets.
- iii. True.
- iv. False: 2 is in C, but not in B.
- v. False: B contains numbers that are not in C.
- vi. False: A does not contain sets of numbers, so no set is an element of A. However, we could say that $C \subseteq A$.
- vii. True.
- viii. False: The number 1 is in B, and there is no number in C you can subtract from it that gives a result that is non-negative.

Solution 2.2. We have to show that

$$\big(A \cup B\big) \cup C \subseteq A \cup \big(B \cup C\big) \qquad \text{and} \qquad A \cup \big(B \cup C\big) \subseteq \big(A \cup B\big) \cup C.$$

We work out the first of these inclusions; the other is very similar.

Let $x \in \big(A \cup B\big) \cup C$. Then there are two (non-mutually exclusive) possibilities:

- $x \in A \cup B$, so $x \in A$ or $x \in B$. If $x \in A$, then certainly $x \in A \cup \big(B \cup C\big)$. Otherwise, $x \in B$ so $x \in B \cup C$, so again $x \in A \cup \big(B \cup C\big)$.
- $x \in C$, in which case $x \in B \cup C$, so $x \in A \cup \big(B \cup C\big)$.

In all cases we concluded that $x \in A \cup \big(B \cup C\big)$.

Solution 2.3. General hints:

- To prove that $A = B$ you usually need to prove $A \subseteq B$ and $B \subseteq A$.

- To prove that $A \subseteq B$, start with $x \in A$ as a premise and try to prove that $x \in B$. Use the definition of subset to conclude $A \subseteq B$.
- One way to prove set theorems is to rewrite the theorem in logic form and then prove the logic form e.g. $A \subseteq B$ is True if and only if $\forall x \in \mathcal{U} \quad [(x \in A) \Rightarrow (x \in B)]$ is True.

Hints for specific subquestions:

(a) Use the axiom $(x \in B) \vee \neg(x \in B)$.

(c) You may need to use the logical equivalence $[p \wedge (q \vee r)] \equiv [(p \wedge q) \vee (p \wedge r)]$.

Solution 2.4. If $x \in S_1 \cap S_2$, then we have simultaneously that $x \in A$, $x \notin B$, $x \in A$, and $x \in B$, contradiction. So $S_1 \cap S_2 = \varnothing$. The same argument shows that $S_3 \cap S_2 = \varnothing$.

If $x \in S_1 \cap S_2$, then $x \in A$, $x \notin B$, $x \in B$, and $x \notin A$, contradiction.

The union is

$$S_1 \cup S_2 \cup S_3 = A \cup B.$$

To show this, first note that if $C \subseteq X$ and $D \subseteq Y$ then $C \cup D \subseteq X \cup Y$. Since $S_1 = A \setminus B \subseteq A$ and $S_2 = A \cap B \subseteq A$, we have that $S_1 \cup S_2 \subseteq A$. Now $S_3 = B \setminus A \subseteq B$, so $(S_1 \cup S_2) \cup S_3 \subseteq A \cup B$.

For the inclusion in the other direction, let $x \in A \cup B$. Suppose first that $x \in A$. Either $x \in B$ or $x \notin B$, leading to $x \in A \setminus B = S_1$ or $x \in A \cap B = S_2$, so that $x \in S_1 \cup S_2 \subseteq S_1 \cup S_2 \cup S_3$.

Similarly, if $x \in B$ we conclude that $x \in S_2 \cup S_3$, so in either case $x \in S_1 \cup S_2 \cup S_3$.

Solution 2.5.

(a) The length is 1.

(b) $C_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$. The length is 2/3.

(c) $C_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{1}{3}] \cup [\frac{2}{3}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$. The length is 4/9.

(d)

(e) 0, 1, 1/3, 2/3, 1/9, 2/9, 7/9, 8/9 are all in the set. So is $(1/3)^n$ for all $n \in \mathbf{N}$.

None of the points in the open interval $(1/3, 2/3)$ are in the set. Neither are the points in $(1/9, 2/9)$ and $(7/9, 8/9)$.

Solution 2.6. As the hint indicates, we should think of f and g as subsets of $A \times B$.

Suppose $f = g$.

Let $x \in A$. Then $\big(x, f(x)\big) \in f$. Since $f = g$, we have $\big(x, f(x)\big) \in g$. But we also have that $\big(x, g(x)\big) \in g$. However, g is a function, so by definition, given $x \in A$ there is a unique $y \in B$ such that $(x, y) \in g$. This means that $f(x) = g(x)$.

As $x \in A$ was arbitrary, we conclude that $f(x) = g(x)$ for all $x \in A$.

Conversely, suppose $f(x) = g(x)$ for all $x \in A$. We claim that $f = g$ as subsets of $A \times B$.

Let $(x, y) \in f$, then $x \in A$ and $y = f(x)$. But $f(x) = g(x)$, so $(x, y) = \big(x, g(x)\big) \in g$. Therefore $f \subseteq g$.

In a similar way, one shows that $g \subseteq f$.

Solution 2.7.

(a) Suppose f is injective.

Let $y \in B$ and suppose $x_1, x_2 \in A$ satisfy $f(x_1) = y$ and $f(x_2) = y$. Then $f(x_1) = f(x_2)$. By the definition of injectivity, this implies that $x_1 = x_2$. Hence the equation $y = f(x)$ has at most one solution (since we proved that any two solutions must be equal).

Conversely, suppose $\forall y \in B$, the equation $f(x) = y$ has at most one solution $x \in A$.

Let $x_1, x_2 \in A$ be such that $f(x_1) = f(x_2)$. Let $y = f(x_1)$, then $y \in B$ and x_1, x_2 are both solutions to the equation $f(x) = y$, so by assumption we have $x_1 = x_2$. Hence f is injective.

(b) Suppose f is surjective.

Let $y \in B$. By the definition of surjectivity, there exists $x \in A$ such that $f(x) = y$. In other words, there is at least one solution for the equation $f(x) = y$.

Conversely, suppose $\forall y \in B$, the equation $f(x) = y$ has at least one solution $x \in A$. Let $y \in B$. Let $x \in A$ be a solution of $f(x) = y$, then of course we have $f(x) = y$. So f is surjective.

(c) This is a direct consequence of Parts (a) and (b).

Solution 2.8. In one direction, suppose f is invertible and let g be its inverse.

To show that f is injective, suppose $f(a_1) = f(a_2)$. Apply g on both sides:

$$a_1 = g(f(a_1)) = g(f(a_2)) = a_2,$$

so $a_1 = a_2$.

To show that f is surjective, let $b \in B$. Let $a = g(b) \in A$. Then

$$f(a) = f(g(b)) = b.$$

In the other direction, suppose f is bijective. By Exercise 2.7 (c) we know that for every $b \in B$, the equation $f(a) = b$ has a unique solution $a \in A$.

We want to define $g : B \longrightarrow A$. Given $b \in B$, let $a \in A$ be the unique solution to $f(a) = b$. Set $g(b) = a$.

We show that g is the inverse of f. It is clear from the definition of g that, for any $a \in A$, we have $g(f(a)) = a$. Now let $b \in B$, and let $a \in A$ be the unique solution to $f(a) = b$. Then by the definition of g, $g(b) = a$, therefore $f(g(b)) = f(a) = b$.

Solution 2.9.

(a) There are no such functions: suppose $f: X \longrightarrow \varnothing$ is a function and let $x \in X$ (such x exists since X is nonempty). Then $f(x) \in \varnothing$, contradicting the fact that the empty set does not have any elements.

(b) There is a unique function $f: \varnothing \longrightarrow X$. It doesn't do anything.

(c) There is a unique function $f: X \longrightarrow \{y\}$: for any $x \in X$ we are forced to let $f(x) = y$, which gives a uniquely determined function.

(d) There are as many functions $f: \{y\} \longrightarrow X$ as there are elements in X. Given $x \in X$, setting $f_x(y) = x$ gives a function $f_x: \{y\} \longrightarrow X$. Clearly if $x_1 \neq x_2$ then $f_{x_1} \neq f_{x_2}$.

Conversely, if $f: \{y\} \longrightarrow X$ is a function, then $f(y) \in X$. If we let $x = f(y)$, then $f = f_x$ as defined above.

If I wrote this more carefully, the argument would make it clear that there is a bijection between the set X and the set of all functions $\{y\} \longrightarrow X$.

Solution 2.10.

(a) I claim that f must be injective. Suppose $a, a' \in A$ such that $f(a) = f(a')$. Then

$$(g \circ f)(a) = g(f(a)) = g(f(a')) = (g \circ f)(a').$$

Since $g \circ f$ is injective, we must have $a = a'$.

There is nothing we can say about the function g. For a counterexample showing that g need not be injective, we can take $f: \mathbf{R} \longrightarrow \mathbf{R}$ given by $f(x) = e^x$ and $g: \mathbf{R} \longrightarrow \mathbf{R}$ given by $g(x) = x^2$.

(b) I claim that g must be surjective. Suppose $c \in C$. Since $g \circ f: A \longrightarrow C$ is surjective, there exists $a \in A$ such that $g(f(a)) = c$. Let $b = f(a) \in B$. Then $g(b) = g(f(a)) = c$.

There is nothing we can say about the function f. For a counterexample showing that f need not be surjective, we can take $f: \mathbf{R} \longrightarrow \mathbf{R}$ given by $f(x) = 0$ and $g: \mathbf{R} \longrightarrow \{1\}$ given by $g(x) = 1$.

(c) We can only conclude what parts (a) and (b) tell us, namely that f is injective and g is surjective.

Solution 2.11. Let $h(a,b) = \big(f(a), g(b)\big)$.

We check that h is injective: suppose $h(a_1,b_1) = h(a_2,b_2)$, so that $f(a_1) = f(a_2)$ and $g(b_1) = g(b_2)$. Since f and g are injective, we get $a_1 = a_2$ and $b_1 = b_2$, so that $(a_1,b_1) = (a_2,b_2)$.

To see that h is surjective, consider an arbitrary element $(c,d) \in C \times D$. Then $c \in C$; since f is surjective, there exists $a \in A$ such that $f(a) = c$. Also $d \in D$; since g is surjective, there exists $b \in B$ such that $g(b) = d$. Then $(a,b) \in A \times B$ and $h(a,b) = \big(f(a), g(b)\big) = (c,d)$.

Solution 2.12.

(a) First we show that $f(X \cup Y) \subseteq f(X) \cup f(Y)$. If $b \in f(X \cup Y)$, then there exists $a \in X \cup Y$ such that $b = f(a)$. So we have $a \in X$ with $b = f(a)$, so that $b \in f(X)$, or $a \in Y$ with $b = f(a)$, so that $b \in f(Y)$. In any case, $b \in f(X) \cup f(Y)$.

Next we show that $f(X) \cup f(Y) \subseteq f(X \cup Y)$. If $b \in f(X) \cup f(Y)$, then $b \in f(X)$ or $b \in f(Y)$. In the first case, we have that $b = f(a)$ for some $a \in X$; in the second case, we have that $b = f(a)$ for some $a \in Y$. In any of the cases, we have $b = f(a)$ for some $a \in X \cup Y$, so that $b \in f(X \cup Y)$.

(b) If $b \in f(X \cap Y)$ then $b = f(a)$ for some $a \in X \cap Y$. Since $a \in X$, we see that $b \in f(X)$; and since $a \in Y$, we see that $b \in f(Y)$, so we conclude that $b \in f(X) \cap f(Y)$.

The opposite inclusion does not always hold. (Follow-up exercise for you: try to prove the opposite inclusion, and identify where the "proof" fails to be correct.) For a counterexample, take $f : \mathbf{R} \longrightarrow \mathbf{R}$ given by $f(x) = x^2$, $X = [1,2]$ and $Y = [-2,-1]$. Then $f(X) = [1,4] = f(Y)$, so that $f(X) \cap f(Y) = [1,4]$, but $X \cap Y = \varnothing$ so that $f(X \cap Y) = \varnothing$.

Solution 2.13.

(a) If $a \in f^{-1}(X \cup Y)$ then $f(a) \in X \cup Y$, so $f(a) \in X$ or $f(a) \in Y$, which means that $a \in f^{-1}(X)$ or $a \in f^{-1}(Y)$, so that $a \in f^{-1}(X) \cup f^{-1}(Y)$.

If $a \in f^{-1}(X) \cup f^{-1}(Y)$, then $a \in f^{-1}(X)$ or $a \in f^{-1}(Y)$, so $f(a) \in X$ or $f(a) \in Y$, which means that $f(a) \in X \cup Y$, so that $a \in f^{-1}(X \cup Y)$.

(b) If $a \in f^{-1}(X \cap Y)$ then $f(a) \in X \cap Y$, so $f(a) \in X$ and $f(a) \in Y$, which means that $a \in f^{-1}(X)$ and $a \in f^{-1}(Y)$, so that $a \in f^{-1}(X) \cap f^{-1}(Y)$.

The opposite inclusion always holds: if $a \in f^{-1}(X) \cap f^{-1}(Y)$, then $a \in f^{-1}(X)$ and $a \in f^{-1}(Y)$, so $f(a) \in X$ and $f(a) \in Y$, hence $f(a) \in X \cap Y$, so that $a \in f^{-1}(X \cap Y)$. (Follow-up exercise: why doesn't the counterexample from the proof of Exercise 2.12(b) translate into a counterexample here?)

Solution 2.14.

(a) Suppose f is injective and let $X \subseteq A$.

If $a \in f^{-1}(f(X))$ then $f(a) \in f(X)$ so there exists $x \in X$ such that $f(a) = f(x)$; since f is injective, this implies that $a = x$, so in particular $a \in X$. Therefore $f^{-1}(f(X)) \subseteq X$.

If $x \in X$ then $f(x) \in f(X)$ so $x \in f^{-1}(f(X))$; in other words $X \subseteq f^{-1}(f(X))$.

Conversely, suppose $f^{-1}(f(X)) = X$ for all $X \subseteq A$ but f is not injective. Then there exist $a_1, a_2 \in A$ such that $f(a_1) = f(a_2)$ but $a_1 \neq a_2$. Let $X = \{a_1\}$ and $b = f(a_1) = f(a_2)$, then $f(X) = \{f(a_1)\} = \{b\}$, but $\{a_1, a_2\} \subseteq f^{-1}(\{b\}) = f^{-1}(f(X))$, so certainly $f^{-1}(f(X)) \neq X$, contradiction.

(b) Suppose f is surjective and let $X \subseteq B$.

If $b \in f(f^{-1}(X))$ then there exists $a \in f^{-1}(X)$ such that $f(a) = b$, but then $f(a) \in X$, so $b \in X$. We conclude that $f(f^{-1}(X)) \subseteq X$.

Now suppose $x \in X$. Since f is surjective, there exists $a \in A$ such that $f(a) = x$. This means that $a \in f^{-1}(X)$, and then that $x \in f(f^{-1}(X))$, so we conclude that $X \subseteq f(f^{-1}(X))$.

Conversely, suppose $f(f^{-1}(X)) = X$ for all $X \subseteq B$. Take an arbitrary $b \in B$ and define $X = \{b\}$. If $f^{-1}(\{b\}) = \varnothing$, then $f(f^{-1}(\{b\})) = f(\varnothing) = \varnothing \neq \{b\}$, contradiction. So we must have that $f^{-1}(\{b\}) \neq \varnothing$ for all $b \in B$, which means that f is surjective.

Solution 2.15.

(a) Since B is nonempty, it contains at least one element $b_0 \in B$. For any $a \in A$, we have $\pi_A(a, b_0) = a$, so π_A is surjective.

The proof for π_B is similar.

(b) We are asked to look for $h : X \longrightarrow A \times B$ such that for all $x \in X$ we have

$$\pi_A(h(x)) = f(x),$$
$$\pi_B(h(x)) = g(x).$$

By the definition of π_A and π_B, this forces us to take

$$h(x) = \big(f(x), g(x)\big).$$

This function h satisfies the required property, and is unique.

Solution 2.16.

(a) $\sup S = 3$ and $\inf S = -3$. Both are in S.

(b) $\sup S = 5$ and $\inf S = -1$. Neither is in S.

(c) $\sup S = 2$ and $\inf S = -3$. Neither is in S.

(d) $\sup S = 0$ and $\inf S = -1$. Neither is in S.

(e) No supremum or infimum.

(f) $\sup S = \sqrt{7}$ and $\inf S = -\sqrt{7}$. Neither is in S.

Solution 2.17. We prove (a) and (f), the others are very similar.

(a) Since S is bounded above, there exists $\alpha \in \mathbf{R}$ such that for all $x \in S$ we have $x \leqslant \alpha$. Let $\beta = c + \alpha \in \mathbf{R}$. Let $y \in c + S$, then there exists $x \in S$ such that $y = c + x$. Therefore

$$y = c + x \leqslant c + \alpha = \beta.$$

Since $y \leqslant \beta$ for all $y \in c + S$, we conclude that $c + S$ is bounded above.

We claim that $c + \sup S$ is the supremum of $c + S$. Since $\sup S$ is an upper bound for S, the argument we just gave shows that $c + \sup S$ is an upper bound for $c + S$. Now let $\beta \in \mathbf{R}$ be any upper bound for $c + S$. So for any $x \in S$ we have $c + x \leqslant \beta$, that is $x \leqslant \beta - c$, so $\beta - c$ is an upper bound for S. Therefore $\sup S \leqslant \beta - c$, hence $c + \sup S \leqslant \beta$.

We conclude that $c + \sup S$ is the least upper bound of $c + S$.

(f) Since S is bounded below, there exists $\alpha \in \mathbf{R}$ such that for all $x \in S$ we have $\alpha \leqslant x$. Then $c\alpha \geqslant cx$ for all $x \in S$, implying that $c\alpha$ is an upper bound for cS. Hence cS is bounded above.

We claim that $c \inf S$ is the supremum of cS. Since $\inf S$ is a lower bound for S, $c \inf S$ is an upper bound for cS. Now let $\beta \in \mathbf{R}$ be any upper bound for cS. So for any $x \in S$ we have $cx \leqslant \beta$, that is $x \geqslant \beta/c$, so β/c is a lower bound for S. Therefore $\inf S \geqslant \beta/c$, hence $c \inf S \leqslant \beta$.

We conclude that $c \inf S$ is the least upper bound of cS.

Solution 2.18. Let $L \subseteq \mathbf{R}$ be the set of lower bounds of B. Since B is bounded below, it has a lower bound ℓ such that $\ell \leqslant b$ for all $b \in B$. So $\ell \in L$, and more importantly L is non-empty. B is also non-empty, so let some $b \in B$. Then any lower bound $\ell \in L$ satisfies $\ell \leqslant b$. Hence L is bounded above by b. By the Completeness Axiom, L is a non-empty subset of $\mathbf{R}$ that is bounded above, so it has a supremum. Define $\ell_0 = \sup L$. We claim that ℓ_0 is the infimum of B.

By the definition of supremum, if ℓ_0 is a lower bound, then it is the greatest lower bound and hence the infimum of B. So we only need to check that ℓ_0 is indeed a lower bound. Suppose not, and that there exists a $b \in B$ such that $b < \ell_0$. Then this b is also an upper bound for L which is less than ℓ_0, contradicting that ℓ_0 is the supremum of L.

Alternative approach: Given a non-empty bounded below set $B \subseteq \mathbf{R}$, let $C = -S = \{-x : x \in B\}$ as in Exercise 2.17.

It is clear that C is non-empty, and it is bounded above by Exercise 2.17 (f). By the Completeness Axiom it has a supremum $s = \sup C$. Therefore by Exercise 2.17 (f), B has an infimum and $\inf B = -\sup C$.

Solution 2.19.

(a) True

(b) True

(c) True

(d) True only if $c > 0$. What if $c = 0$ or $c < 0$?

(e) Sometimes True (Hint: $a < 0 < b$ vs $0 < a < b$)

(f) Sometimes True

(g) True only if $a > 0$.

(h) False. What should it be?

Solution 2.20.

(a) $x \in [-\frac{5}{2}, \frac{3}{2}]$

(b) $x \in (-\infty, -7] \cup [3, \infty)$

(c) $x \in (2, \infty)$

(d) $x \in (-2, 5)$

(e) $x \in (-1, 0)$.

Solution 2.21.

(a) $-3 < x < 3$

(b) $-3 < x < 7$

(c) $1 < x < 2$

(d) $-2 \leqslant x \leqslant 1$

(e) $x \leqslant -7$ or $x \geqslant 3$

(f) $\frac{1}{6} < x < \frac{1}{4}$

(g) $x > 2$

(h) $1 \leqslant x \leqslant \sqrt{3}$ or $-\sqrt{3} \leqslant x \leqslant -1$

Solution 2.22.

(a) $(-\infty, -4] \cup [-2, \infty)$

(b) $(-1, 5)$

(c) $(-1, 5) \cup (-2, 0) = (-2, 5)$

(d) $(-1, 5) \cap (-2, 0) = (-1, 0)$

(e) $[-4, -1)$

(f) $(-\infty, 0] \cup (1, \infty)$.

Solution 2.23. We can get a reasonable bound by estimating an upper bound for the numerator and a lower bound for the denominator.

(a) The numerator is a positive parabola, so taking $x = 1$ gives an upper bound $2 \cdot 1^2 + 1 = 3$. For the denominator, x ranges from -1 to 1, so $x + 3$ ranges from 2 to 4, so taking the lower bound gives 2, and the overall upper bound for f is $\frac{3}{2}$.

(b) The range of x is -3 to 1. The denominator $10 - x^3$ is decreasing on this region so its infimum is 9. The numerator is increasing on this inverval, so its supremum is 5. Hence a reasonable upper bound for f would be $\frac{5}{9}$.

Solution 2.24. By Theorem 2.57, we need to show that

$$-|a - b| \leqslant |a| - |b| \leqslant |a - b|.$$

One application of the triangle inequality gives

$$|a| = |(a - b) + b| \leqslant |a - b| + |b| \qquad \Rightarrow \qquad |a| - |b| \leqslant |a - b|.$$

Another application gives

$$|b| = |(b - a) + a| \leqslant |b - a| + |a| = |a - b| + |a| \qquad \Rightarrow \qquad -|a - b| \leqslant |a| - |b|.$$

Solution 2.25. We know that absolute values are multiplicative, so

$$\left|\frac{a + \varepsilon}{b + \varepsilon}\right| = \frac{|a + \varepsilon|}{|b + \varepsilon|}.$$

By the triangle inequality, one has

$$|a+\varepsilon| \leqslant |a| + |\varepsilon| = |a| + \varepsilon,$$

and

$$|b| = |(b+\varepsilon) - \varepsilon| \leqslant |b+\varepsilon| + |\varepsilon| = |b+\varepsilon| + \varepsilon,$$

which is equivalent to $|b| - \varepsilon \leqslant |b+\varepsilon|$. Since $\varepsilon \leqslant a, b$, there are no sign changes. Putting it all together gives

$$\left|\frac{a+\varepsilon}{b+\varepsilon}\right| = \frac{|a+\varepsilon|}{|b+\varepsilon|} \leqslant \frac{|a|+\varepsilon}{|b|-\varepsilon},$$

as required.

Solution 2.26.

(a) Consider $a = 0$. The right hand side will be positive. Repeat for $b = 0$.

(b) Remember to check the second derivative to make sure it is a minimum rather than a maximum.

(c) Consider $x = b^{\frac{1}{p-1}}$ and $q = p/(p-1)$ and use a bit of algebra to simplify.

(d) Replace x with a and, with a small amount of rearranging, voilà!

Solution 2.27.

(a) Since $x < y$, we use axiom (RO3) to add $-x$ on both sides and get $0 < y - x$.

Since $z < 0$, we use axiom (RO3) to add $-z$ on both sides and get $0 < -z$.

Now we multiply these two positive numbers using axiom (RO4), getting $0 < (y-x)(-z) = xz - yz$.

Adding yz to both sides using axiom (RO3), we have $yz < xz$.

The other statement is the special case $z = -1$.

(b) Hint: Otherwise $1/x < 0$. (Why?) Multiply both sides by x.

(c) We start with $x < y$ and add x to both sides using axiom (RO3), so that $2x < x+y$. Now we apply Theorem 2.41 with $z = 1/2$ to get

$$x = 2x\,\frac{1}{2} = (x+y)\,\frac{1}{2} = \frac{x+y}{2}.$$

To get the other inequality in the statement, add y to both sides of $x + y$.

Solution 2.28.

(a) If $x < 0$, then $|x| = -x$ so $-x \leqslant |x|$. Also $x < 0$ implies $x < 0 \leqslant |x|$, so $x \leqslant |x|$.

If $x \geqslant 0$, then $|x| = x$ so $x \leqslant |x|$. Also if $x \geqslant 0$ implies $x \geqslant 0 \geqslant -|x|$, so $-x \leqslant |x|$.

(b) By Theorem 2.57 we have

$$|x - y| < \varepsilon \qquad \text{if and only if} \qquad -\varepsilon < x - y < \varepsilon,$$

now add y everywhere.

Solution 2.29.

(a) $x + 1 = 0$.

(b) $2x = 1$.

(c) $x^2 = 2$.

Solution 2.30. Use the Archimedean Principle III with $y = 1$ and $z = \varepsilon$. One obtains that that for all $0 < \varepsilon$, there exists an $n \in \mathbf{N}$ such that $1 < n\varepsilon$. Divide through by n to obtain

$$(\forall \varepsilon > 0)[(\exists n \in \mathbf{N}) 1/n < \varepsilon],$$

as required.

Solution 2.31.

(a) Let $\beta = 2$. I claim that $x \leqslant 2$ for all $x \in A$.

To prove this, I proceed by contradiction. Suppose there exists $x \in A$ such that $x > 2$. Then $x^2 > 4$, so $x \notin A$, contradiction.

(b) The set A is non-empty since $1 \in A$: $1^2 = 1 \leqslant 2$.

It is bounded above by Part (a), so by the Completeness Axiom it has a supremum.

(c) i. As scrap work, we have that

$$(\alpha + 1/n)^2 = \alpha^2 + \frac{2\alpha}{n} + \frac{1}{n^2} < 2.$$

Since $\frac{1}{n^2} < \frac{1}{n}$, we could require

$$\alpha^2 + \frac{2\alpha}{n} + \frac{1}{n} < 2 \quad \Rightarrow \quad \frac{2\alpha + 1}{2 - \alpha^2} < n.$$

Proof. By the Archimedean Principle, there exists $n \in \mathbf{Z}_{\geqslant 1}$ such that $\frac{2\alpha+1}{2-\alpha^2} < n$. Hence

$$\begin{aligned}
\frac{2\alpha+1}{2-\alpha^2} &< n \\
\frac{2\alpha+1}{n} &< 2-\alpha^2 \\
\alpha^2 + \frac{2\alpha}{n} + \frac{1}{n} &< 2 \\
\alpha^2 + \frac{2\alpha}{n} + \frac{1}{n^2} &< 2 \qquad \text{(since } n \geqslant 1\text{)} \\
(\alpha + 1/n)^2 &< 2.
\end{aligned}$$

ii. By Part (c)i., $(\alpha+1/n)^2 < 2$. Hence $\alpha + 1/n \in A$. Since $\alpha + 1/n \in A$ and $\alpha + 1/n > \alpha$, α cannot be an upper bound of A.

(d) Similar to Part (c).

(e) Since $1 \in A$ and $\alpha = \sup A$, $0 < 1 \leqslant \alpha$. By (O1), either $\alpha^2 < 2$, $\alpha^2 > 2$, or $\alpha^2 = 2$. Hence by Parts (c) and (d), $\alpha^2 = 2$.

Solution 2.32.

(a)
- Given $x \in A$, we have $f(x) = f(x)$ so $x \sim x$.
- If $x \sim y$, then $f(x) = f(y)$, so $f(y) = f(x)$, that is $y \sim x$.
- If $x \sim y$ and $y \sim z$ then $f(x) = f(y)$ and $f(y) = f(z)$, so that $f(x) = f(z)$, that is $x \sim z$.

(b)
- Given $k \in \mathbf{Z}$, $k - k = 0$ is divisible by n.
- If $k \sim m$, then $m - k = na$ for some $a \in \mathbf{Z}$, therefore $k - m = -na$, so $m \sim k$.
- If $k \sim m$ and $m \sim \ell$ then $m - k = na$ and $\ell - m = nb$ for some $a, b \in \mathbf{Z}$. Therefore $\ell - k = n(a+b)$ so $k \sim \ell$.

(c)
- Given $X \in A$, the identity function $\mathrm{id}_X \colon X \longrightarrow X$ is bijective, so $X \sim X$.
- If $X \sim Y$ then there is a bijective function $f \colon X \longrightarrow Y$, so there's a bijective inverse function $f^{-1} \colon Y \longrightarrow X$, that is $Y \sim X$.
- If $X \sim Y$ and $Y \sim Z$, then there are bijective functions $f \colon X \longrightarrow Y$ and $g \colon Y \longrightarrow Z$. The composition $g \circ f \colon X \longrightarrow Z$ is bijective, so $X \sim Z$.

(d) Let $x, z \in A$. There are two possibilities:
- $x \sim z$: given $y \in [x]$, we have $x \sim y$, so $y \sim x$, so $y \sim z$, so $y \in [z]$. This tells us that $[x] \subseteq [z]$, and the other inclusion follows the same way from $z \sim x$. Therefore $[x] = [z]$.

- $x \not\sim z$: suppose $[x] \cap [z]$ is not empty, and pick some element y in there. Then $y \in [x]$ so $y \sim x$, and $y \in [z]$ so $y \sim z$, implying that $x \sim z$, contradiction. Therefore $[x] \cap [z] = \varnothing$.

(e) Given $m \in \mathbf{Z}$, let $0 \leqslant r \leqslant n-1$ be the remainder of the division of m by n: $m = qn + r$. Then $m - r$ is divisible by n, hence $m \sim r$. From the previous part, we know that there are at most n equivalence classes, one for each possible value of r. To show that we have exactly n equivalence classes, we need to prove that $[r_1] \neq [r_2]$ for any $r_1 \neq r_2$ with $0 \leqslant r_1, r_2 \leqslant n-1$. We do this by contradiction: if $[r_1] = [r_2]$ then $r_1 \sim r_2$, so $r_2 - r_1$ is a multiple of n. But $-(n-1) \leqslant r_2 - r_1 \leqslant (n-1)$, and the only multiple of n in that interval is 0, in other words $r_2 = r_1$, contradiction.

(f) Suppose π is bijective. I claim that the only way $x \sim y$ can happen is if $x = y$: if $x \sim y$ then $\pi(x) = \pi(y)$, but π is bijective so $x = y$.
We conclude that the equivalence relation on A must be given by: $x \sim y$ if and only if $x = y$.

(g) i.
- Given $(a, b) \in \mathbf{N} \times \mathbf{N}$, we have $a + b = b + a$ so $(a, b) \sim (a, b)$.
- If $(a, b) \sim (c, d)$ then $a + d = b + c$, so $c + b = d + a$, that is $(c, d) \sim (a, b)$.
- If $(a, b) \sim (c, d)$ and $(c, d) \sim (x, y)$ then $a + d = b + c$ and $c + y = d + x$. Adding these two equalities gives $a+d+c+y = b+c+d+x$, and cancelling out $c + d$ on both sides we get $a + y = b + x$, that is $(a, b) \sim (x, y)$.

ii. Define $g\colon B \longrightarrow \mathbf{Z}$ by $g\big([(a, b)]\big) = a - b$. We first need to make sure that this is a well-defined function, in other words that the value does not depend on the chosen representative (a, b) of $[(a, b)]$: suppose $(a', b') \in [(a, b)]$, then $(a', b') \sim (a, b)$ so $a' + b = b' + a$, hence $a' - b' = a - b$.

Let's show that g is injective: if $g([(a, b)]) = g([(c, d)])$ then $a - b = c - d$, so $a + d = b + c$, so $(a, b) \sim (c, d)$, so $[(a, b)] = [(c, d)]$.

Finally, to see that g is surjective, let $n \in \mathbf{Z}$. If $n \geqslant 0$ then $n = g([(n + 1, 1)])$; if $n < 0$ then $n = g([(1, 1 - n)])$.