MAST20026 Assignment 3

Due Thursday 14 May at 8pm (aka 20:00) on Canvas and GradeScope

Some guidelines:

- Your answers to this assignment can be handwritten (on physical paper and scanned, or on a tablet or other device), or typeset using a system that can produce professional-quality mathematical documents (e.g. LaTeX or Typst, but not Microsoft Word).

  If you are writing by hand, make sure that your writing can appear clearly enough on the document you upload to Gradescope. This is usually achieved by writing legibly with a very readable writing implement.

- Please indicate clearly which question you are writing about at the top of each page. (Ideally, start a new question on a new page.)

  When you upload your document to Gradescope, please mark which pages correspond to which questions.

- The quality of the exposition will be assessed alongside the correctness of the approach.

  There is no need to include your preparatory scratch work (do this on separate paper) but make sure that the solution you submit is a complete explanation.

  "Completeness" of the explanation is somewhat subjective, but: results from the lectures, tutorials, exercises can be used (without having to re-prove them). Make sure you say clearly what result(s) you are using, though.

- It is acceptable for students to discuss the questions on the assignments and strategies for solving them. However, each student must write down their solutions in their own words and notation (and make sure that they understand what they are writing).

- As a large language model, I do not have an opinion about your use of generative AI to complete this assignment.

  Actually... I do have an opinion.

  Whatever resource you tap into, use it in a smart way: know its limitations, and do the work of really understanding what it is that you are submitting. This is true of your mate who is smart but tends to make arithmetic mistakes, of your favourite linear algebra or analysis book that uses completely different notation to ours, or of the chatbot that sounds impressive but hallucinates references or gives you a proof that relies on lots of results we have not seen in the subject (and that's the best case scenario). Do your job: be paranoid, double-check everything, take it apart and put it back together until it makes sense to you. Why? See the next point.

- Assignments are a valuable learning tool in this subject, so strive to maximise their impact on your understanding of the material.

- No Chegg or anything similar. At all. Please.

This assignment consists of 5 questions. Please scan your answer pages and upload them to GradeScope in the correct order.

As usual, make sure that you reference (by name or number) any result that you are using from lectures, tutorials, self-study problems, past assignments, etc.

3.1. (8 marks)

(a) Find the limit of the sequence (x_n) given by

$$x_n = \frac{-2n^2 + 4n + \sin(n)}{8n^2 + 4n + 2}, \quad n \in \mathbf{N}, n \geqslant 1.$$

(You may use without proof the fact that $|\sin(x)| \leqslant 1$ for all $x \in \mathbf{R}$.)

(b) Let (y_n) be the sequence defined by

$$y_n = \sqrt[2026]{\log_2(n)}, \quad n \in \mathbf{N}, n \geqslant 1.$$

Prove that (y_n) diverges.

Aim for a proof that is short but complete. In this part, there will be marks assigned for conciseness.

3.2. (8 marks)

Consider the sequence (c_n) given by

$$c_n = \cos(n), \qquad n \in \mathbf{N}.$$

(a) Prove that there exists a subsequence (c_{n_k}) of (c_n) with

$$c_{n_k} \geqslant \frac{1}{\sqrt{2}} \qquad \text{for all } k \in \mathbf{N}.$$

(b) Prove that (c_n) diverges.

(c) Prove that (c_n) has a convergent subsequence.

(d) **Not for credit, but an obvious question:** Can you **explicitly** construct a convergent subsequence of (c_n)?

3.3. (7 marks)

Let (b_n) be the sequence defined by

$$b_0 = 1 \quad \text{and} \quad b_{n+1} = \frac{1}{2 + b_n} \quad \text{for all } n \geqslant 0.$$

(a) Prove that $0 < b_n < \frac{1}{2}$ for all $n \geqslant 1$.

(b) Prove that the sequence (b_n) is contractive.

(c) Find the limit of the sequence (b_n).

3.4. (12 marks)

In each part of this question, you are asked to give an example of a particular type of mathematical object.

For each part, you should either: (i) write down a specific example and prove that it satisfies the required properties; or (ii) prove that no object with the required properties exists.

(a) Functions $f, g : \mathbf{R} \longrightarrow \mathbf{R}$ such that f and g are both discontinuous at 1, but the product fg is continuous at 1.

(b) A finite subset $E \subseteq \mathbf{R}$ such that the set of limit points of E is $\{5\}$.

(c) A proper nonempty subset $E \subsetneq \mathbf{R}$ such that the set of limit points of E is E.

3.5 (The Struggle is $\mathcal{R}$eal, Episode II)**. (14 marks)**

You should probably have Episode I (that is, Assignment Question 2.5) handy, as this builds upon the work done there.

We denote the element $[z_n] \in \mathcal{R}$ found in Assignment Question 2.5(g) by 0 and the element $[u_n]$ by 1, so we have $0, 1 \in \mathcal{R}$ and

$$0 + [y_n] = [y_n] \qquad \text{and} \qquad 1[y_n] = [y_n] \qquad \text{for all } [y_n] \in \mathcal{R}.$$

Let $(x_n), (y_n) \in \mathsf{Seq}(\mathbf{Q})$ be Cauchy sequences. Consider the statement

$$p\big((x_n), (y_n)\big) : \ (\exists \delta \in \mathbf{Q}_{>0})(\exists N \in \mathbf{N}) n > N \Rightarrow x_n + \delta < y_n.$$

(a) Prove that if $(x'_n), (y'_n) \in \mathsf{Seq}(\mathbf{Q})$ are such that $(x_n) \heartsuit (x'_n)$ and $(y_n) \heartsuit (y'_n)$, then

$$p\big((x_n), (y_n)\big) \Rightarrow p\big((x'_n), (y'_n)\big).$$

Define a relation $<$ on $\mathcal{R}$ as follows:

$$[x_n] < [y_n] \text{ if } p\big((x_n), (y_n)\big) \text{ is } \mathsf{True}.$$

(b) Prove that: for all $[x_n], [y_n], [z_n] \in \mathcal{R}$, if $[x_n] < [y_n]$ and $[y_n] < [z_n]$, then $[x_n] < [z_n]$.

(c) Prove that: for all $[x_n], [y_n], [c_n] \in \mathcal{R}$, if $[x_n] < [y_n]$ then $[x_n] + [c_n] < [y_n] + [c_n]$.

(d) Prove that: for all $[x_n], [y_n] \in \mathcal{R}$, if $0 < [x_n]$ and $0 < [y_n]$ then $0 < [x_n][y_n]$.

(e) Let p, q, r be logical statements and consider the statement S given by "exactly one of p, q, r is True".

Express S as the conjunction of several logical statements $S_1 \wedge S_2 \wedge \cdots \wedge S_k$ such that each S_j uses only the connectives $\vee$, $\neg$, $\wedge$.

Verify your work by writing out the truth table of S.

(f) Consider the statement T given by

T : "For all $[x_n], [y_n] \in \mathcal{R}$, exactly one of the following is True:

$$[x_n] < [y_n], \qquad [y_n] < [x_n], \qquad [x_n] = [y_n]."$$

Suppose you want to prove that the statement T is True. Use your work in part (e) to divide the task of proving T into several simpler mathematical statements. (You are not asked to prove these simpler statements, only to list them.)