

MAST20026 Assignment 4

Due Wednesday 27 May at 8pm (aka 20:00) on Canvas and GradeScope

Some guidelines:

- Your answers to this assignment can be handwritten (on physical paper and scanned, or on a tablet or other device), or typeset using a system that can produce professional-quality mathematical documents (e.g. \LaTeX or Typst, but not Microsoft Word).

If you are writing by hand, make sure that your writing can appear clearly enough on the document you upload to Gradescope. This is usually achieved by writing legibly with a very readable writing implement.

- Please indicate clearly which question you are writing about at the top of each page. (Ideally, start a new question on a new page.)

When you upload your document to Gradescope, please mark which pages correspond to which questions.

- The quality of the exposition will be assessed alongside the correctness of the approach.

There is no need to include your preparatory scratch work (do this on separate paper) but make sure that the solution you submit is a complete explanation.

“Completeness” of the explanation is somewhat subjective, but: results from the lectures, tutorials, exercises can be used (without having to re-prove them). Make sure you say clearly what result(s) you are using, though.

- It is acceptable for students to discuss the questions on the assignments and strategies for solving them. However, each student must write down their solutions in their own words and notation (and make sure that they understand what they are writing).

- As a large language model, I do not have an opinion about your use of generative AI to complete this assignment.

Actually... I do have an opinion.

Whatever resource you tap into, use it in a smart way: know its limitations, and do the work of really understanding what it is that you are submitting. This is true of your mate who is smart but tends to make arithmetic mistakes, of your favourite linear algebra or analysis book that uses completely different notation to ours, or of the chatbot that sounds impressive but hallucinates references or gives you a proof that relies on lots of results we have not seen in the subject (and that's the best case scenario). Do your job: be paranoid, double-check everything, take it apart and put it back together until it makes sense to you. Why? See the next point.

- Assignments are a valuable learning tool in this subject, so strive to maximise their impact on your understanding of the material.

- No Chegg or anything similar. At all. Please.

This assignment consists of 3 questions. Please scan your answer pages and upload them to GradeScope in the correct order.

4.1 (Series Bingo). Complete the following table:

Series	Converges? (Y/N)	Brief reason
(a) $\sum_{k=3}^{\infty} \frac{1}{k \log(k) \log(\log(k))^2}$		
(b) $\sum_{k=3}^{\infty} (-1)^k \frac{\log(k)}{k}$		
(c) $\sum_{k=1}^{\infty} \left(\sqrt[3]{k^3 + k} - k \right)$		
(d) $\sum_{k=2}^{\infty} \frac{2 + \sin(k!)}{k^2 + \cos(k)}$		
(e) $\sum_{k=1}^{\infty} \frac{k! 3^k}{k^k}$		

4.2 (Examples Bingo). Give an example of each of the following, or state that such a request is impossible. For any that are impossible, supply a short explanation for why this is the case.

Request	Exists? (Y/N)	Explicit example OR brief reason why impossible
(a) Continuous $f : (0, 1) \rightarrow \mathbf{R}$ and Cauchy sequence (x_n) such that $(f(x_n))$ is not Cauchy		
(b) Continuous $f : [0, \infty) \rightarrow \mathbf{R}$ and Cauchy sequence (x_n) such that $(f(x_n))$ is not Cauchy		
(c) Differentiable $f : [0, 1] \rightarrow \mathbf{R}$ that does not attain its minimum on $[0, 1]$		
(d) Differentiable $f : [-1, 1] \rightarrow \mathbf{R}$ such that $f' : [-1, 1] \rightarrow \mathbf{R}$ is not differentiable at 0		
(e) Non-constant integrable $f : [0, 1] \rightarrow \mathbf{R}$ such that $\int_0^1 f(x) dx = 0$		

4.3 (The Struggle is \mathcal{R} al, Episode III). (You may want to quickly review the highlights of Episodes I and II from Assignments 2 and 3. And now, for the season finale...)

Your hard work of helping Rationalians establish a wider set of numbers \mathcal{R} to enhance their traditional set \mathbf{Q} is starting to pay off. But there are those who cling to the old ways and maintain that

- there was nothing wrong or missing in their beloved \mathbf{Q} ;
- or if there was, we have no evidence that the newfangled \mathcal{R} is free of these shortcomings;
- and regardless of any improvements, the new numbers \mathcal{R} are just too much of a departure from the familiar and comfortable \mathbf{Q} .

Your final task is to counteract the third of these objections as follows:

Prove that \mathbf{Q} is dense in \mathcal{R} : for any $[x_n], [y_n] \in \mathcal{R}$ with $[x_n] < [y_n]$, there exists $r \in \mathbf{Q}$ such that

$$[x_n] < \iota(r) < [y_n].$$

Hence, \mathbf{Q} and \mathcal{R} are, in fact, (arbitrarily) close to each other.

[Aside/Prologue: It remains to deal with the first two objections.

- For the first, you can deduce from [Exercise 2.31](#) that the supremum α of the set

$$A = \{x \in \mathbf{Q} : x^2 \leq 2\}$$

satisfies $\alpha^2 = 2$, hence $\alpha \notin \mathbf{Q}$. The conclusion is that the ordered set \mathbf{Q} does not satisfy the Completeness Axiom, since there exists a non-empty subset A of \mathbf{Q} that is bounded above in \mathbf{Q} but does not have a supremum in \mathbf{Q} .

- The way to deal with the second objection is to prove that \mathcal{R} satisfies the Completeness Axiom: “every non-empty subset of \mathcal{R} that is bounded above in \mathcal{R} has a supremum in \mathcal{R} .” Unfortunately, our production budget for this season is exhausted (as are we), and we have to postpone this until a future season (perhaps a cameo on the Metric and Hilbert Spaces show?)

That’s it!]