MAST20026 Assignment 1

Due Thursday 26 March at 8pm (aka 20:00) on Canvas and GradeScope

Some guidelines:

- Your answers to this assignment can be handwritten (on physical paper and scanned, or on a tablet or other device), or typeset using a system that can produce professional-quality mathematical documents (e.g. LaTeX or Typst, but not Microsoft Word).

  If you are writing by hand, make sure that your writing can appear clearly enough on the document you upload to Gradescope. This is usually achieved by writing legibly with a very readable writing implement.

- Please indicate clearly which question you are writing about at the top of each page. (Ideally, start a new question on a new page.)

  When you upload your document to Gradescope, please mark which pages correspond to which questions.

- The quality of the exposition will be assessed alongside the correctness of the approach.

  There is no need to include your preparatory scratch work (do this on separate paper) but make sure that the solution you submit is a complete explanation.

  "Completeness" of the explanation is somewhat subjective, but: results from the lectures, tutorials, exercises can be used (without having to re-prove them). Make sure you say clearly what result(s) you are using, though.

- It is acceptable for students to discuss the questions on the assignments and strategies for solving them. However, each student must write down their solutions in their own words and notation (and make sure that they understand what they are writing).

- As a large language model, I do not have an opinion about your use of generative AI to complete this assignment.

  Actually... I do have an opinion.

  Whatever resource you tap into, use it in a smart way: know its limitations, and do the work of really understanding what it is that you are submitting. This is true of your mate who is smart but tends to make arithmetic mistakes, of your favourite linear algebra or analysis book that uses completely different notation to ours, or of the chatbot that sounds impressive but hallucinates references or gives you a proof that relies on lots of results we have not seen in the subject (and that's the best case scenario). Do your job: be paranoid, double-check everything, take it apart and put it back together until it makes sense to you. Why? See the next point.

- Assignments are a valuable learning tool in this subject, so strive to maximise their impact on your understanding of the material.

- No Chegg or anything similar. At all. Please.

This assignment consists of 4 questions. Please scan your answer pages and upload them to GradeScope in the correct order.

1.1. (11 marks) Let p, q be statements and consider the compound statement:

$$S: \ [(\neg p) \Rightarrow q] \Rightarrow [p \Rightarrow (\neg q)].$$

(a) Use a truth table to determine whether S is a tautology, a contradiction, or neither. Briefly explain your reasoning by referring to specific lines of your truth table.

(b) If p is False and S is True, what can you deduce about the truth value of q? Briefly explain your reasoning by referring to specific lines of your truth table from Part (a).

(c) Write down $\neg S$, the negation of S; simplify the answer.

(d) Write down a statement T that is logically equivalent to $\neg S$ and uses as few connectives $(\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow)$ as possible. (You can use as many parentheses as needed.)

You should justify why the number of connectives that you have used is minimal.

Solution.

(a) Here is the truth table:

p	q	$\neg p$	$\neg q$	$\neg p \Rightarrow q$	$p \Rightarrow (\neg q)$	$[(\neg p) \Rightarrow q] \Rightarrow [p \Rightarrow (\neg q)]$
T	T	F	F	T	F	F
T	F	F	T	T	T	T
F	T	T	F	T	T	T
F	F	T	T	F	T	T

The statement is neither a tautology nor a contradiction.

Since S is False in the first line of the truth table but True in the rest, S is neither a tautology nor contradiction.

(b) The statement q could be either False or True.

The last two lines of the truth table above where p is False and S is True.

On the third line, q is True, and on the fourth line, q is False.

(c) Recall from the lecture slides (1-31) that $\neg(p \Rightarrow q) \equiv p \wedge (\neg q)$.

We use this to negate S:

$$\neg[(\neg p \Rightarrow q) \Rightarrow (p \Rightarrow \neg q)] \equiv (\neg p \Rightarrow q) \wedge \neg(p \Rightarrow \neg q) \equiv (\neg p \Rightarrow q) \wedge (p \wedge q).$$

(d) I claim that $\neg S$ is equivalent to $T: \ p \wedge q$. Here is a truth table verifying this:

p	q	$\neg p$	$\neg q$	$\neg p \Rightarrow q$	$p \wedge q$	$(\neg p \Rightarrow q) \wedge (p \wedge q)$
T	T	F	F	T	T	T
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	F	F

Why is T minimal? It has exactly one connective. Therefore the only possibility for T to not be minimal is if T is equivalent to an expression in p and q that has no connectives, but there are only four possible such expressions: p, q, True, and False.

We see in the truth table that T is not equivalent to any of these four expressions.

We conclude that the minimum number of connectives is one. □

1.2. (11 marks) Consider the condition:

$$p(x, y): \; xy \in \mathbf{Z}.$$

(a) Translate the statement $(\forall x \in \mathbf{Q})\ [(\exists y \in \mathbf{Q})\ p(x, y)]$ into an English sentence.

(b) Is the statement $(\forall x \in \mathbf{Q})\ [(\exists y \in \mathbf{Q})\ p(x, y)]$ True or False? Briefly explain.

(c) Is the statement $(\exists y \in \mathbf{Q})\ (y \neq 0) \wedge [(\forall x \in \mathbf{Q})\ p(x, y)]$ True or False? Briefly explain.

(d) Translate the statement below into an English sentence and determine whether it is True or False. Briefly explain.

$$(\forall x, y, z \in \mathbf{Q})\ [p(x, y) \wedge p(y, z) \Rightarrow p(x, z)].$$

Solution.

(a) For every rational number x there exists a rational number y such that xy is an integer.

(b) The statement is True as we explain now.
Let $x \in \mathbf{Q}$. Write $x = \frac{m}{n}$, where $m, n \in \mathbf{Z}$, $n \neq 0$.
Let $y = n$, then $xy = m \in \mathbf{Z}$, so $p(x, y)$ is True.

(c) The statement is False; here is a brief argument that shows that the statement leads to a contradiction.
Suppose there exists $y \in \mathbf{Q}$ with the desired properties: $y \neq 0$ and $(\forall x \in \mathbf{Q})\ xy \in \mathbf{Z}$.

- If $y \notin \mathbf{Z}$, let $x = 1$, then $xy = y \notin \mathbf{Z}$, contradiction.
- If $y \in \mathbf{Z}$, let $x = \frac{1}{2y}$, then $xy = \frac{1}{2} \notin \mathbf{Z}$, contradiction.

(d) An English sentence could be:

For any rational numbers x, y and z, if xy and yz are integers, then xz is an integer.

This is False, as the following counterexample shows:
Let $x, y, z \in \mathbf{Q}$ with $x = \frac{1}{2} = z, y = 2$. Then, we have $xy = yz = 1 \in \mathbf{Z}$. But $xz = \frac{1}{4} \notin \mathbf{Z}$.
(Many other counterexamples can be found.) □

1.3. (8 marks) Give a **formal proof** of the following statement:

"Let x be an integer. The integer $x^2 + 4x + 7$ is odd if and only if x is even."

Solution. Let $x \in \mathbf{Z}$ and suppose $x^2 + 4x + 7$ is odd.

$p_1(x): \; (\exists k \in \mathbf{Z}) x^2 + 4x + 7 = 2k + 1$	(hypothesis and def of odd)
$p_2(x): \; (\exists k \in \mathbf{Z}) x^2 = 2(-2x + k - 3)$	($p_1(x)$ and algebra)
$p_3(x): \; x^2$ is even	($p_2(x)$ and def of even)
$p_4(x): \; x$ is even	($p_3(x)$ and Tutorial Question 3.3).

We conclude that if $x^2 + 4x + 7$ is odd, then x is even.

In the other direction: let $x \in \mathbf{Z}$ and suppose x is even.

$p_1(x): \ (\exists m \in \mathbf{Z})x = 2m$	(hypothesis and def of even)
$p_2(x): \ (\exists m \in \mathbf{Z})x^2 + 4x + 7 = 2(2m^2 + 4m + 3) + 1$	($p_1(x)$ and algebra)
$p_3(x): \ x^2 + 4x + 7$ is odd	($p_2(x)$ and def of odd).

We conclude that if x is even, then $x^2 + 4x + 7$ is odd. □

1.4. (12 marks) Consider the statement

$$S: \quad \text{“There exist integers } x \text{ and } k \text{ such that } x^2 = 4k + 3.\text{”}$$

(a) Write S in the language of formal logic and mathematics.

(b) Write the negation $\neg S$ in the language of formal logic and mathematics (give a simplified answer, that is, move the negation as far to the right as possible in the statement).

(c) Write the negation $\neg S$ in English.

(d) Prove the negation $\neg S$. (Preferably a rigorously justified **informal proof**, but you may give a formal proof if you want to.)

You may use without proof the fact that 4 does not divide the integers 1, 2, 3.

(e) Given integers a and b such that $a > b \geqslant 0$, consider the statement

$$S(a,b): \quad \text{“There exist integers } x \text{ and } k \text{ such that } x^2 = ak + b.\text{”}$$

In the previous parts you considered the case $S = S(4,3)$ and proved that it is False.

Now, find:

i. one other pair (a,b) such that $S(a,b)$ is False;
ii. a pair (a,b) such that $S(a,b)$ is True.

(You are not asked to prove that these pairs work.)

Solution.

(a) $(\exists x, k \in \mathbf{Z})x^2 = 4k + 3$.

(b) $(\forall x, k \in \mathbf{Z})x^2 \neq 4k + 3$.

(c) For all integers x and k we have $x^2 \neq 4k + 3$. Or: there do not exist integers x and k such that $x^2 = 4k + 3$.

(d) Suppose $x, k \in \mathbf{Z}$ exist such that $x^2 = 4k + 3$.

- If x is even, then there exists $\ell \in \mathbf{Z}$ such that $x = 2\ell$. Then

$$x^2 = 4\ell^2, \qquad \text{so} \qquad 4\ell^2 = 4k + 3, \qquad \text{so} \qquad 4(\ell^2 - k) = 3.$$

This implies that 4 divides 3, which is a contradiction.

- If x is odd, then there exists $\ell \in \mathbf{Z}$ such that $x = 2\ell + 1$. Then

$$x^2 = 4\ell^2 + 4\ell + 1, \qquad \text{so} \qquad 4\ell^2 + 4\ell + 1 = 4k + 3, \qquad \text{so} \qquad 4(\ell^2 + \ell - k) = 2.$$

  This implies that 4 divides 2, which is a contradiction.

In all possible cases we have obtained a contradiction, so we conclude that such integers x, k do not exist.

(e) There are many examples for each of the two scenarios.

$S(3, 2)$ and $S(4, 2)$ are False.

$S(a, 0)$ is True for all $a > 0$; $S(a, 1)$ is True for all $a > 1$; $S(7, 2)$ is True.

(We say that b is a *quadratic residue modulo a* if $S(a, b)$ is True. There is a lot of fun mathematics involving these.) □