MAST20026 Assignment 2

Due Monday 20 April at 8pm (aka 20:00) on Canvas and GradeScope

Some guidelines:

- Your answers to this assignment can be handwritten (on physical paper and scanned, or on a tablet or other device), or typeset using a system that can produce professional-quality mathematical documents (e.g. LaTeX or Typst, but not Microsoft Word).

  If you are writing by hand, make sure that your writing can appear clearly enough on the document you upload to Gradescope. This is usually achieved by writing legibly with a very readable writing implement.

- Please indicate clearly which question you are writing about at the top of each page. (Ideally, start a new question on a new page.)

  When you upload your document to Gradescope, please mark which pages correspond to which questions.

- The quality of the exposition will be assessed alongside the correctness of the approach.

  There is no need to include your preparatory scratch work (do this on separate paper) but make sure that the solution you submit is a complete explanation.

  "Completeness" of the explanation is somewhat subjective, but: results from the lectures, tutorials, exercises can be used (without having to re-prove them). Make sure you say clearly what result(s) you are using, though.

- It is acceptable for students to discuss the questions on the assignments and strategies for solving them. However, each student must write down their solutions in their own words and notation (and make sure that they understand what they are writing).

- As a large language model, I do not have an opinion about your use of generative AI to complete this assignment.

  Actually... I do have an opinion.

  Whatever resource you tap into, use it in a smart way: know its limitations, and do the work of really understanding what it is that you are submitting. This is true of your mate who is smart but tends to make arithmetic mistakes, of your favourite linear algebra or analysis book that uses completely different notation to ours, or of the chatbot that sounds impressive but hallucinates references or gives you a proof that relies on lots of results we have not seen in the subject (and that's the best case scenario). Do your job: be paranoid, double-check everything, take it apart and put it back together until it makes sense to you. Why? See the next point.

- Assignments are a valuable learning tool in this subject, so strive to maximise their impact on your understanding of the material.

- No Chegg or anything similar. At all. Please.

This assignment consists of 5 questions. Please scan your answer pages and upload them to GradeScope in the correct order.

2.1. (8 marks) Let X and Y be sets and let $g: X \longrightarrow Y$ be a function. Recall the notions of *(direct) image* under g of a subset $A \subseteq X$:

$$g(A) = \{y \in Y : (\exists x \in A) y = g(x)\},$$

and of *inverse image* (or *preimage*) under g of a subset $B \subseteq Y$:

$$g^{-1}(B) = \{x \in X : g(x) \in B\}.$$

(a) Prove that for any subset $B \subseteq Y$ we have

$$g^{-1}\big(Y \setminus B\big) = X \setminus g^{-1}(B).$$

(b) Consider the statements

$$(*) \qquad \text{for any subset } A \subseteq X \text{ we have} \quad g(X \setminus A) \subseteq Y \setminus g(A)$$

and

$$(**) \qquad \text{for any subset } A \subseteq X \text{ we have} \quad Y \setminus g(A) \subseteq g(X \setminus A).$$

Show that taking $X = Y = \{-1, 0, 1\}$, $A = \{-1\}$, and $g: X \longrightarrow Y$ defined by $g(x) = x^2$, gives a counterexample to both statements (*) and (**).

(c) What additional assumption(s) on the function g are necessary to make statement (*) True? Give a proof.

(d) What additional assumption(s) on the function g are necessary to make statement (**) True? Give a proof.

Solution.

(a) We prove the equality by showing that each set is a subset of the other.

First, let $x \in g^{-1}\big(Y \setminus B\big)$. By the definition of preimage, $g(x) \in Y \setminus B$. This means $g(x) \notin B$. Hence (by the definition of preimage) $x \notin g^{-1}(B)$. Therefore $x \in X \setminus g^{-1}(B)$. This shows $g^{-1}\big(Y \setminus B\big) \subseteq X \setminus g^{-1}(B)$.

Conversely, let $x \in X \setminus g^{-1}(B)$. Then $x \notin g^{-1}(B)$. By the definition of preimage, this means $g(x) \notin B$. Thus $g(x) \in Y \setminus B$. Therefore $x \in g^{-1}\big(Y \setminus B\big)$. Hence $X \setminus g^{-1}(B) \subseteq g^{-1}\big(Y \setminus B\big)$.

(b) We have

$$X \setminus A = \{0, 1\}, \qquad g(X \setminus A) = \{0, 1\}, \qquad g(A) = \{1\}, \qquad Y \setminus g(A) = \{-1, 0\},$$

which disproves both statements.

(c) It is enough to know that $g: X \longrightarrow Y$ is an injective function.

Let $y \in g(X \setminus A)$, then there exists $x \in X \setminus A$ such that $y = g(x)$. Suppose that $y \in g(A)$, then there exists $x' \in A$ such that $y = g(x')$. Since g is injective and $g(x') = g(x)$, we conclude that $x' = x$, which is a contradiction since $x' \in A$ and $x \notin A$. So our assumption that $y \in g(A)$ was False, hence $y \in Y \setminus g(A)$.

(d) It is enough to know that $g: X \longrightarrow Y$ is a surjective function.

Let $y \in Y \setminus g(A)$. Since g is surjective, there exists $x \in X$ such that $y = g(x)$. Suppose that $x \in A$, then $y \in g(A)$, contradicting the fact that $y \notin g(A)$. Therefore $x \notin A$, so $y = g(x) \in g(X \setminus A)$. $\square$

2.2. (7 marks) Consider the set $S = \{n \in \mathbf{Z} \mid n \geqslant 1\}$. Define a function $f:\ S \longrightarrow \mathbf{Z}$ by

$$f(n) = \begin{cases} \dfrac{n}{2}, & \text{if } n \text{ is even,} \\ -\dfrac{n+1}{2}, & \text{if } n \text{ is odd.} \end{cases}$$

(a) Determine whether f is injective. Justify your answer.

(b) Determine whether f is surjective. Justify your answer.

Solution.

(a) For injectivity: suppose $f(a) = f(b)$ for some $a, b \in S$.

We consider the three cases:

- a and b are both even.
  Then
  $$f(a) = \frac{a}{2}, \qquad f(b) = \frac{b}{2}.$$
  If $f(a) = f(b)$, then
  $$\frac{a}{2} = \frac{b}{2} \Rightarrow a = b.$$
- a and b are both odd.
  Then
  $$f(a) = -\frac{a+1}{2}, \qquad f(b) = -\frac{b+1}{2}.$$
  If $f(a) = f(b)$, then
  $$-\frac{a+1}{2} = -\frac{b+1}{2} \Rightarrow a+1 = b+1 \Rightarrow a = b.$$
- One of a, b is even and the other is odd. Without loss of generality, suppose a is even and b is odd.
  Since a is even, $f(a) = a/2 \geqslant 1$. Since b is odd, $f(b) = -(b+1)/2 \leqslant -1$. Hence $f(a) \neq f(b)$, contradiction, so this case cannot actually occur.

We conclude that $f(a) = f(b)$ implies $a = b$, so f is injective.

(b) For surjectivity: we need to determine whether for every $z \in \mathbf{Z}$, there exists $n \in S$ such that $f(n) = z$.

I claim that 0 is not in the image of f. We proceed by contradiction: suppose there exists $n \in S$ such that $f(n) = 0$.

There are two possibilities:

- n is even.
  Then $f(n) = n/2$, but then $0 = n/2$ implies that $n = 0$, contradicting the fact that $n \in S$.
- n is odd.
  Then $f(n) = -(n+1)/2$, but then $0 = -(n+1)/2$ implies that $n = -1$, contradicting the fact that $n \in S$.

Since all cases lead to a contradiction, we conclude that 0 is not in the image of f, so f is not surjective. □

2.3. (8 marks)
Given subsets $A, B \subseteq \mathbf{R}$, define

$$A + B = \{a + b :\ a \in A \text{ and } b \in B\}.$$

Suppose that A and B are nonempty and bounded above.

(a) Prove that $A + B$ is nonempty.

(b) Let $s = \sup A$ and $t = \sup B$. Prove that $s + t$ is an upper bound for $A + B$.

(c) Now let u be an arbitrary upper bound for $A + B$, and let $a \in A$. Show that $t \leqslant u - a$.

(d) Finally, show that

$$\sup(A + B) = s + t = \sup A + \sup B.$$

Solution.

(a) Since A is nonempty, there exists some element $a_0 \in A$. Since B is nonempty, there exists some element $b_0 \in B$. Then $a_0 + b_0 \in A + B$, showing that $A + B$ is nonempty.

(b) Take any element $a + b \in A + B$ with $a \in A$ and $b \in B$. Because $s = \sup A$, we have $a \leqslant s$. Because $t = \sup B$, we have $b \leqslant t$.

Adding these inequalities gives

$$a + b \leqslant s + t.$$

Hence every element of $A + B$ is less than or equal to $s + t$, so $s + t$ is an upper bound for $A + B$.

(c) Let u be an arbitrary upper bound for $A + B$ and fix $a \in A$.

For every $b \in B$, we have $a + b \in A + B$. Since u is an upper bound,

$$a + b \leqslant u.$$

Subtracting a from both sides gives

$$b \leqslant u - a$$

for all $b \in B$. Hence $u - a$ is an upper bound for B. Since $t = \sup B$, we obtain

$$t \leqslant u - a.$$

(d) From part (b), $s + t$ is an upper bound for $A + B$, so

$$\sup(A + B) \leqslant s + t.$$

Now let u be any upper bound for $A + B$. From part (c), for every $a \in A$ we have

$$t \leqslant u - a,$$

which implies

$$a + t \leqslant u \quad \Rightarrow \quad a \leqslant u - t.$$

This inequality holds for all $a \in A$, so $u - t$ is an upper bound of A. Since s is the supremum of A we get

$$s \leqslant u - t \quad \Rightarrow \quad s + t \leqslant u.$$

Thus $s + t$ is less than or equal to every upper bound u of $A + B$. Therefore $s + t$ is the least upper bound of $A + B$, and

$$\sup(A + B) = s + t. \qquad \square$$

2.4. (8 marks) Compute the supremum and infimum of each of the following sets:

(a) $A = \{1 + m/n : \; m, n \in \mathbf{N} \text{ with } m < n\}$.

(b) $B = \{(-1)^m/n : \; m, n \in \mathbf{N}, n \neq 0\}$.

(c) $C = \{n/(5n+1) : \; n \in \mathbf{N}\}$.

(d) $D = \{m/(m+n) : \; m, n \in \mathbf{N}, m + n \neq 0\}$.

Give a brief explanation for each of your findings.

Solution.

(a) Since $m < n$, we have $0 \leqslant m/n < 1$. Hence $1 \leqslant 1 + m/n < 2$.

Values of m/n can be arbitrarily close to 1 from below (e.g., $m = n - 1$ and $n \longrightarrow \infty$), so $1 + m/n$ can be arbitrarily close to 2 from below.

Also when $m = 0$, $m/n = 0$, and $1 + m/n = 1$.

Therefore, $\inf A = 1$, $\sup A = 2$.

(b) If m is even, $(-1)^m = 1$ and the elements are $1/n$. If m is odd, $(-1)^m = -1$ and the elements are $-1/n$.

Thus

$$B = \left\{ \frac{1}{n}, -\frac{1}{n} : \; n \in \mathbf{N}, n \neq 0 \right\}.$$

The largest value occurs at $n = 1$: $\sup B = 1$. The smallest value occurs at $n = 1$: $\inf B = -1$.

(c) For all $n \in \mathbf{N}, n \neq 0$, we have $n/(5n+1) \leqslant n/5n = 1/5$. If $n = 0$, the inequality still holds, that $0/1 = 0 < 1/5$.

All values are less than $1/5$, but approach $1/5$ as $n \longrightarrow \infty$.

When $n = 0$, we obtain $0/1 = 0$, and for all $n \neq 0$, $n/(5n+1) > 0$. Therefore, 0 is the minimal.

Thus $\sup C = 1/5$ and $\inf C = 0$.

(d) We have $0 \leqslant m/(m+n) \leqslant 1$, for any $m, n \in D$.

When $m = 0$, and $n = 1$, we obtain $0/1 = 0$, and when $m = 1, n = 0$, we obtain $1/1 = 1$.

Thus, $\inf D = 0$, $\sup D = 1$. $\square$

2.5 (The Struggle is $\mathcal{R}$eal, Episode I)**. (17 marks)**
You wake up on planet Rationalia, where the real numbers have not (yet) been discovered. The rational numbers $\mathbf{Q}$ are, however, well-known. Their properties appear in the famous Scroll of Rationalia, reproduced on the following page.
You decide to help the Rationalians in their quest to $\mathcal{R}$eally revolutionise their mathematics. But you may not use anything about the real numbers $\mathbf{R}$ in your work. (You don't want them to find out you're an alien, do you?)

(a) Prove that $1/(n+1) \longrightarrow 0$, and that for all $L \in \mathbf{Q}$, the constant sequence (L) converges to L.

We say that two sequences $(x_n), (x'_n) \in \mathsf{Seq}(\mathbf{Q})$ are *associated*, written $(x_n) \heartsuit (x'_n)$, if $x_n - x'_n \longrightarrow 0$ as $n \longrightarrow \infty$.

(b) Let

$$(x_n) = \left(1 + \frac{n}{n+1}\right), \qquad (y_n) = \left(2 - \frac{1}{(n+1)^2}\right), \qquad (z_n) = \left(\frac{3}{n+1}\right).$$

Prove that (x_n) is associated to (y_n), but that (x_n) is not associated to (z_n).

(c) Prove that if $(x_n) \heartsuit (x'_n)$ and (x_n) is Cauchy, then (x'_n) is Cauchy.

Given a sequence $(x_n) \in \mathsf{Seq}(\mathbf{Q})$, we define its *association* $[x_n]$ to be the set of all its associates:

$$[x_n] = \big\{(x'_n) \in \mathsf{Seq}(\mathbf{Q}) : \ (x'_n) \heartsuit (x_n)\big\}.$$

Finally, we let $\mathcal{R}$ be the set of all associations of Cauchy sequences:

$$\mathcal{R} = \big\{[x_n] : \ (x_n) \in \mathsf{Seq}(\mathbf{Q}) \text{ and } (x_n) \text{ is Cauchy}\big\}.$$

(d) There is a function $\iota : \ \mathbf{Q} \longrightarrow \mathcal{R}$ given by $\iota(x) = [x]$, where $[x]$ is the association of the constant sequence $(x, x, x, x, \dots)$. Prove that ι is an injective function.

(e) Define an operation of addition $+ : \ \mathcal{R} \times \mathcal{R} \longrightarrow \mathcal{R}$ by

$$[x_n] + [y_n] = [x_n + y_n] \qquad \text{for all } [x_n], [y_n] \in \mathcal{R}.$$

Prove that this is a well-defined function, in other words that

i. If (x_n) and (y_n) are Cauchy then $(x_n + y_n)$ is Cauchy.

ii. For all $(x_n), (x'_n), (y_n), (y'_n) \in \mathsf{Seq}(\mathbf{Q})$ we have:

$$\text{if } (x'_n) \heartsuit (x_n) \text{ and } (y'_n) \heartsuit (y_n) \text{ then } (x'_n + y'_n) \heartsuit (x_n + y_n).$$

(f) Taking your inspiration from the previous part, define an operation of multiplication on $\mathcal{R}$, and write down the statements that would imply that this multiplication is well-defined. (You are not asked to prove these statements.)

(g) Find an element $[z_n] \in \mathcal{R}$ such that $[z_n] + [y_n] = [y_n]$ for all $[y_n] \in \mathcal{R}$.

Find an element $[u_n] \in \mathcal{R}$ such that $[u_n][y_n] = [y_n]$ for all $[y_n] \in \mathcal{R}$.

Solution.

(a) Let $\varepsilon \in \mathbf{Q}_{>0}$. Let $x = \frac{1}{\varepsilon} - 1$. If $x \leqslant 0$, take $M = 1 > x$. If $x > 0$, then use the Archimedean Property of $\mathbf{N}$ in $\mathbf{Q}$ to find $M \in \mathbf{N}$ such that $M > x$. If $n > M$, then

$$n > M > x = \frac{1}{\varepsilon} - 1 \Rightarrow n + 1 > \frac{1}{\varepsilon} \Rightarrow \left|\frac{1}{n+1} - 0\right| = \frac{1}{n+1} < \varepsilon.$$

Therefore $1/(n+1) \longrightarrow 0$.

Now consider the constant sequence L, where each term is $x_n = L$. Let $\varepsilon \in \mathbf{Q}_{>0}$. Let $M = 0$. If $n > M$, then

$$|x_n - L| = |L - L| = 0 < \varepsilon.$$

(b) We have

$$x_n - y_n = \frac{n}{n+1} - 1 + \frac{1}{n+1}^{2} = -\frac{1}{n+1} + \frac{1}{(n+1)^2} \longrightarrow -0 + 0^2 = 0,$$

where we used the fact that $\frac{1}{n+1} \longrightarrow 0$ from part (a) and the Algebra of Limits Theorem.

We have

$$x_n - z_n = 1 + \frac{n}{n+1} - \frac{3}{n+1} = \frac{2n-2}{n+1} = 2 - \frac{4}{n+1} \longrightarrow 2 - 0 \neq 0,$$

where we used the fact that $\frac{1}{n+1} \longrightarrow 0$ from part (a) and the Algebra of Limits Theorem.

(Alternatively: use part (a) and the Algebra of Limits to show that $x_n \longrightarrow 2$, $y_n \longrightarrow 2$, and $z_n \longrightarrow 0$, then conclude.)

(c) Let $\varepsilon \in \mathbf{Q}_{>0}$. Since (x_n) is Cauchy, there exists $M_1 \in \mathbf{N}$ such that for all $j, k \in \mathbf{N}$ we have

$$j, k > M_1 \Rightarrow |x_j - x_k| < \frac{\varepsilon}{3}.$$

Since $(x_n) \heartsuit (x'_n)$, there exists $M_2 \in \mathbf{N}$ such that for all $n \in \mathbf{N}$ we have

$$n > M_2 \Rightarrow |x'_n - x_n| < \frac{\varepsilon}{3}.$$

Let $M = \max\{M_1, M_2\}$ and let $j, k \in \mathbf{N}$. Then

$$j, k > M \Rightarrow |x'_j - x'_k| \leqslant |x'_j - x_j| + |x_j - x_k| + |x_k - x'_k| < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.$$

We conclude that (x'_n) is Cauchy.

(d) Suppose $x, y \in \mathbf{Q}$ are such that $\iota(x) = \iota(y)$. Then $[x] = [y]$, that is the constant sequences $(x, x, x, \dots)$ and $(y, y, y, \dots)$ are associates. Therefore the sequence $(x - y, x - y, x - y, \dots)$ converges to 0. But it is a constant sequence, therefore it converges to $x - y$ by part (a). We conclude that $x - y = 0$, so $x = y$. Therefore ι is injective.

(e) Let $\varepsilon \in \mathbf{Q}_{>0}$. Since (x_n) is Cauchy, there exists $M_1 \in \mathbf{N}$ such that for all $j, k \in \mathbf{N}$ we have

$$j, k > M_1 \Rightarrow |x_j - x_k| < \frac{\varepsilon}{2}.$$

Since (y_n) is Cauchy, there exists $M_2 \in \mathbf{N}$ such that for all $j, k \in \mathbf{N}$ we have

$$j, k > M_2 \Rightarrow |y_j - y_k| < \frac{\varepsilon}{2}.$$

Let $M = \max\{M_1, M_2\}$ and let $j, k \in \mathbf{N}$. Then

$$j, k > M \Rightarrow |(x_j + y_j) - (x_k + y_k)| \leqslant |x_j - x_k| + |y_j - y_k| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

We conclude that $(x_n + y_n)$ is Cauchy.

Suppose the sequences (x_n), (x'_n), (y_n), and (y'_n) satisfy the given conditions. We want to show that $(x'_n + y'_n) - (x_n + y_n) \longrightarrow 0$ as $n \longrightarrow \infty$.

Let $\varepsilon \in \mathbf{Q}_{>0}$. Since $(x'_n) \heartsuit (x_n)$, there exists $M_1 \in \mathbf{N}$ such that for all $n \in \mathbf{N}$ we have

$$n > M_1 \Rightarrow |x'_n - x_n| < \frac{\varepsilon}{2}.$$

Since $(y'_n) \heartsuit (y_n)$, there exists $M_2 \in \mathbf{N}$ such that for all $n \in \mathbf{N}$ we have

$$n > M_2 \Rightarrow |y'_n - y_n| < \frac{\varepsilon}{2}.$$

Let $M = \max\{M_1, M_2\}$ and let $n \in \mathbf{N}$, then

$$n > M \Rightarrow |(x'_n + y'_n) - (x_n + y_n)| \leqslant |x'_n - x_n| + |y'_n - y_n| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Therefore $(x'_n + y'_n) \heartsuit (x_n + y_n)$.

(f) Define $\cdot : \mathcal{R} \times \mathcal{R} \longrightarrow \mathcal{R}$ by

$$[x_n] \cdot [y_n] = [x_n y_n] \qquad \text{for all } [x_n], [y_n] \in \mathcal{R}.$$

The two statements are:

- If (x_n) and (y_n) are Cauchy then $(x_n y_n)$ is Cauchy.
- If $(x_n), (x'_n), (y_n), (y'_n) \in \mathsf{Seq}(\mathbf{Q})$ and $(x'_n) \heartsuit (x_n)$ and $(y'_n) \heartsuit (y_n)$ then $(x'_n y'_n) \heartsuit (x_n y_n)$.

(g) Let $z_n = 0$ for all $n \in \mathbf{N}$, in other words $[z_n]$ is the association of the constant sequence $(0, 0, 0, \dots)$.

Let $u_n = 1$ for all $n \in \mathbf{N}$, in other words $[u_n]$ is the association of the constant sequence $(1, 1, 1, \dots)$.

The desired properties now follow immediately from the definitions of addition and multiplication on $\mathcal{R}$. □

The Scroll of Rationalia

The set $\mathbf{Q}$ of rational numbers and its subsets $\mathbf{Z}$ and $\mathbf{N}$ are well-known. Rational numbers can be added, subtracted, multiplied, divided according to the usual rules of arithmetic.

There is an order relation $<$ on $\mathbf{Q}$, in which $0 < 1$.

There is an absolute value function on $\mathbf{Q}$ given by $|\cdot| : \mathbf{Q} \longrightarrow \mathbf{Q}$ where

$$|x| = \begin{cases} -x & \text{if } x < 0, \\ x & \text{otherwise.} \end{cases}$$

It gives rise to a notion of distance $|x - y|$ between any $x, y \in \mathbf{Q}$.

There is a set $\mathsf{Seq}(\mathbf{Q})$ of all sequences of rational numbers, that is

$$\mathsf{Seq}(\mathbf{Q}) = \{x : \mathbf{N} \longrightarrow \mathbf{Q}\} = \big\{(x_n) : (\forall n \in \mathbf{N}) x_n \in \mathbf{Q}\big\}.$$

The following are known to be True. (Some are axioms, some are definitions, some are statements that can be proved from the axioms and the definitions.)

(S1) For all $x, y \in \mathbf{Q}$, exactly one of the following is True:

$$x < y, \qquad y < x, \qquad x = y.$$

(S2) For all $x, y, z \in \mathbf{Q}$, if $x < y$ and $y < z$ then $x < z$.

(S3) For all $x, y, c \in \mathbf{Q}$, if $x < y$ then $x + c < y + c$.

(S4) For all $x, y \in \mathbf{Q}$, if $0 < x$ and $0 < y$ then $0 < xy$.

(S5) The *Archimedean Property of* $\mathbf{N}$ *in* $\mathbf{Q}$: for every $x \in \mathbf{Q}_{>0}$ there exists $n \in \mathbf{N}$ such that $n > x$.

(S6) The *Triangle Inequality*: for every $x, y \in \mathbf{Q}$ we have $|x + y| \leqslant |x| + |y|$.

(S7) A sequence $(x_n) \in \mathsf{Seq}(\mathbf{Q})$ *converges* to some $L \in \mathbf{Q}$ if: for every $\varepsilon \in \mathbf{Q}_{>0}$, there exists $M \in \mathbf{N}$ such that for all $n \in \mathbf{N}$, $n > M$ implies that $|x_n - L| < \varepsilon$.

(S8) A sequence $(x_n) \in \mathsf{Seq}(\mathbf{Q})$ is *bounded* if there exists $C \in \mathbf{Q}$ such that $|x_n| \leqslant C$ for all $n \in \mathbf{N}$.

(S9) A sequence $(x_n) \in \mathsf{Seq}(\mathbf{Q})$ is *Cauchy* if: for every $\varepsilon \in \mathbf{Q}_{>0}$, there exists $M \in \mathbf{N}$ such that for all $m, n \in \mathbf{N}$, $m, n > M$ implies that $|x_m - x_n| < \varepsilon$.

(S10) Every sequence in $\mathsf{Seq}(\mathbf{Q})$ that converges is Cauchy.

(S11) Every Cauchy sequence in $\mathsf{Seq}(\mathbf{Q})$ is bounded.

(S12) The *Algebra of Limits Theorem*: if $(x_n), (y_n) \in \mathsf{Seq}(\mathbf{Q})$ with $x_n \longrightarrow \alpha$ and $y_n \longrightarrow \beta$, where $\alpha, \beta \in \mathbf{Q}$, then:

- $x_n + y_n \longrightarrow \alpha + \beta$;
- $x_n - y_n \longrightarrow \alpha - \beta$;
- $x_n y_n \longrightarrow \alpha\beta$;
- $x_n / y_n \longrightarrow \alpha/\beta$, if $\beta \neq 0$ and $y_n \neq 0$ for all $n \in \mathbf{N}$.