3 Sequences

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3.1 Definition and examples

Recall that, in our convention, the first natural number is 0:

$$\mathbf{N} = \{0, 1, 2, 3, \dots\}.$$

Definition 3.1. A *sequence* (of real numbers) is a function $x : \ \mathbf{N} \longrightarrow \mathbf{R}$.

We usually denote $x(n)$ by x_n and we call it the n-th *term* of the sequence; we denote the sequence itself by $(x_n) = (x_0, x_1, x_2, x_3, \dots)$.

Example 3.2. $(x_n) = (n^2 + 1) = (1, 2, 5, 10, 17, \dots)$.

Sometimes we want to start the sequence at a point $n_0 \in \mathbf{N}$, $n_0 > 0$. In this case we would denote the sequence as

$$\big(x_n\big)_{n \geqslant n_0} = \big(x_{n_0}, x_{n_0+1}, x_{n_0+2}, \dots\big).$$

Example 3.3.

$$\big(x_n\big)_{n\geqslant 1} = \left(\frac{1}{n}\right)_{n\geqslant 1} = \left(1, \frac{1}{2}, \frac{1}{3}, \dots\right).$$

Occasionally people write simply $(1/n)$ for this sequence, and we have to infer that the starting point should be 1 instead of 0.

Example 3.4. We can "approximate" π by the sequence

$$(x_n) = (3, 3.1, 3.14, 3.141, 3.1416, \ldots).$$

Intuitively, the terms of this sequence converge to the real number π.

On the other hand, the sequence $\big((-1)^n\big) = (1, -1, 1, -1, \ldots)$ does not appear to converge.

Neither does the sequence $(2n) = (0, 2, 4, 6, 8, 10, 12, \ldots)$.

We need to make the intuitive notion of "convergence" more precise.

Example 3.5. Consider the sequence

$$(x_n)_{n\geqslant 1} = \left(4 + (-1)^n \frac{2}{\sqrt{n}}\right)_{n\geqslant 1} = \left(4 - \frac{2}{\sqrt{1}}, 4 + \frac{2}{\sqrt{2}}, 4 - \frac{2}{\sqrt{3}}, 4 + \frac{2}{\sqrt{4}}, \dots\right)$$
$$\approx \big(2, 5.41, 2.85, 5, 3.11, 4.82, 3.24, 4.71, \dots\big).$$

Not so clear what's going on? Computing further:

n	x_n	n	x_n
100	4.2	999	3.937
10000	4.02	99999	3.994
1000000	4.002	9999999	3.9994.

Looks like we're tending towards 4 in the long run.

We are trying to express mathematically the situation: "as n gets larger and larger, the term x_n gets closer and closer to 4".

Let's note that this is the same as: "as n gets larger and larger, the distance $|x_n - 4|$ gets smaller and smaller".

3.2 Limit and convergence

Definition 3.6. Let (x_n) be a sequence and let $L \in \mathbf{R}$.

We say that (x_n) *converges to L* if: for every $\varepsilon > 0$ there exists $M \in \mathbf{N}$ such that for all $n \in \mathbf{N}$, if $n > M$ then $|x_n - L| < \varepsilon$.

In this case, we write

$$x_n \longrightarrow L \qquad \text{or} \qquad \lim_{n\to\infty} x_n = L$$

and call L a *limit* of the sequence (x_n).

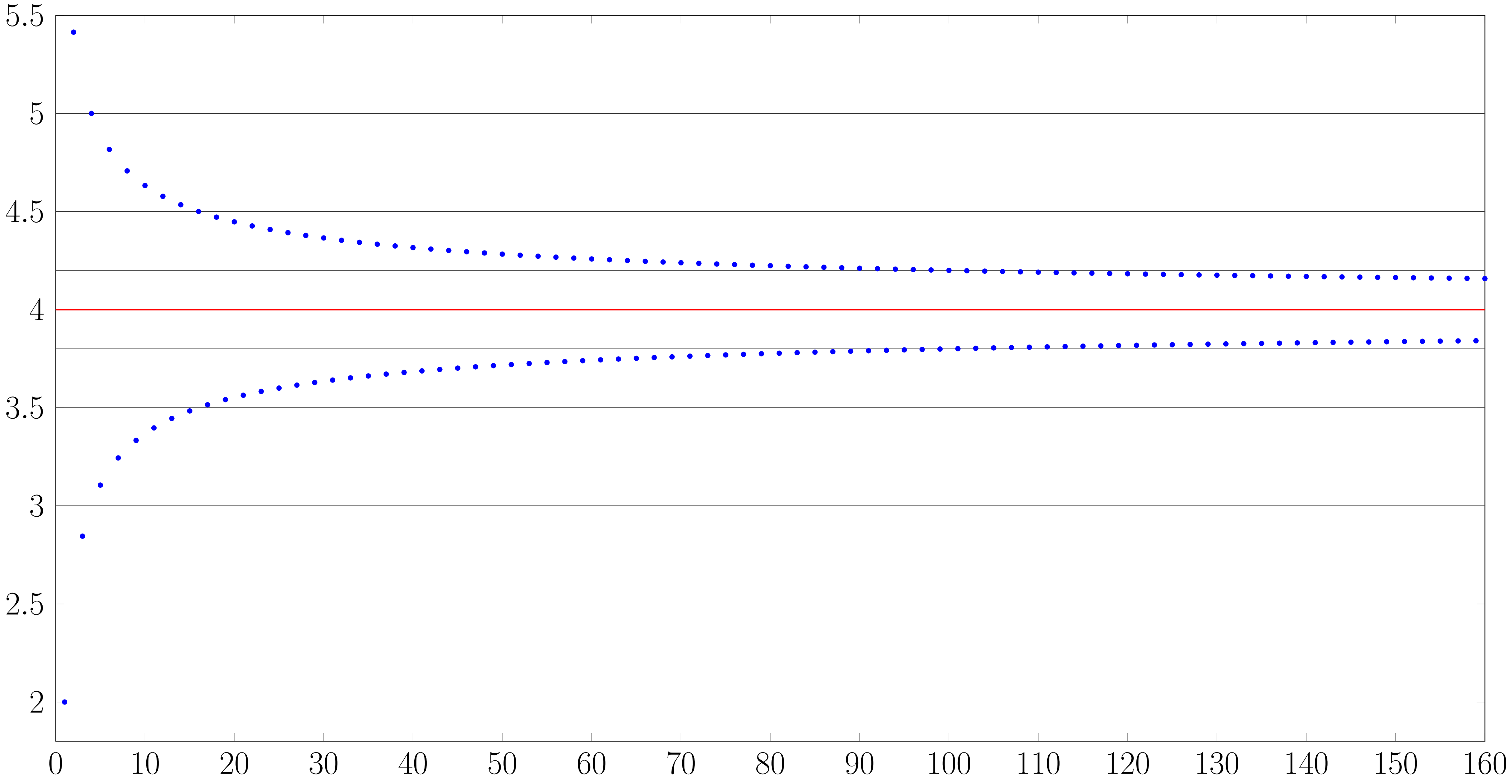
In formal logic notation, the definition of "(x_n) converges to L" is

$$(\forall \varepsilon > 0)(\exists M \in \mathbf{N})(\forall n \in \mathbf{N})\ n > M \Rightarrow |x_n - L| < \varepsilon.$$

Its negation "(x_n) does not converge to L" is

$$(\exists \varepsilon > 0)(\forall M \in \mathbf{N})(\exists n \in \mathbf{N})\ n > M \wedge |x_n - L| \geqslant \varepsilon.$$

Going back to the sequence from Example 3.5:



Example 3.7. Prove that $1 - \dfrac{1}{n} \longrightarrow 1$.

Scrap work. Fix $\varepsilon > 0$ and "solve for" M:

$$\begin{aligned} |x_n - 1| &< \varepsilon \\ \left|1 - \frac{1}{n} - 1\right| &< \varepsilon \\ \left|-\frac{1}{n}\right| &< \varepsilon \\ \frac{1}{n} &< \varepsilon \\ \frac{1}{\varepsilon} &< n. \end{aligned}$$

So: take $M = \lceil 1/\varepsilon \rceil$.

Proof. Let $\varepsilon > 0$. Define $M = \lceil 1/\varepsilon \rceil$. Let $n \in \mathbf{N}$ be such that $n > M$. Then:

$$\begin{aligned} \frac{1}{\varepsilon} &< n \\ \frac{1}{n} &< \varepsilon \\ \left|-\frac{1}{n}\right| &< \varepsilon \\ \left|1 - \frac{1}{n} - 1\right| &< \varepsilon \\ |x_n - 1| &< \varepsilon. \end{aligned}$$

We conclude that $x_n \longrightarrow 1$.

We return to:

Example 3.8. Prove that

$$\left(4 + (-1)^n \frac{2}{\sqrt{n}}\right) \longrightarrow 4.$$

Scrap work. Fix $\varepsilon > 0$ and "solve for" M:

$$\begin{aligned} |x_n - 4| &< \varepsilon \\ \left|(-1)^n \frac{2}{\sqrt{n}}\right| &< \varepsilon \\ \frac{2}{\sqrt{n}} &< \varepsilon \\ \frac{2}{\varepsilon} &< \sqrt{n} \\ \frac{4}{\varepsilon^2} &< n. \end{aligned}$$

So: take $M = \lceil 4/\varepsilon^2 \rceil$.

Proof. Let $\varepsilon > 0$. Define $M = \lceil 4/\varepsilon^2 \rceil$. Let $n \in \mathbf{N}$ be such that $n > M$. Then:

$$\begin{aligned} \frac{4}{\varepsilon^2} &< n \\ \frac{2}{\varepsilon} &< \sqrt{n} \\ \frac{2}{\sqrt{n}} &< \varepsilon \\ \left|(-1)^n \frac{2}{\sqrt{n}}\right| &< \varepsilon \\ |x_n - 4| &< \varepsilon. \end{aligned}$$

We conclude that $x_n \longrightarrow 4$.

Definition 3.9. Let (x_n) be a sequence.

We say that (x_n) *converges* (without reference to a particular limit) if there exists $L \in \mathbf{R}$ such that $x_n \longrightarrow L$.

We say that (x_n) *diverges* if it does not converge.

So (x_n) diverges if for all $L \in \mathbf{R}$, the statement "(x_n) converges to L" is False. In formal logic notation:

$$(\forall L \in \mathbf{R})(\exists \varepsilon > 0)(\forall M \in \mathbf{N})(\exists n \in \mathbf{N})\ n > M \wedge |x_n - L| \geqslant \varepsilon.$$

Example 3.10. Consider the sequence $\big((-1)^n\big) = (1, -1, 1, -1, \dots)$.

Let $x_n = (-1)^n$ for all $n \in \mathbf{N}$.

Suppose that (x_n) converges, so there exists $L \in \mathbf{R}$ such that $x_n \longrightarrow L$.

Let $\varepsilon = 1$. There exists $M \in \mathbf{N}$ such that for all $n > M$ we have $|x_n - L| < 1$.

Let $n > M$ such that n is even. Then $x_n = 1$ so $|x_n - L| = |1 - L| < 1$. Also $n+1$ is odd, so $x_{n+1} = -1$ and $|x_{n+1} - L| = |-1 - L| < 1$.

But by the triangle inequality we have

$$2 = |2| = |(1 - L) - (-1 - L)| \leqslant |1 - L| + |-1 - L| < 1 + 1 = 2,$$

which is a contradiction.

So the sequence (x_n) diverges.

3.3 Bounded sequences

The sequence $(n^2) = (0, 1, 4, 9, 16, \dots)$ also looks like it should diverge, but for a different reason.

Definition 3.11. We say that a sequence (x_n) is *bounded* if the set $\{x_n : \ n \in \mathbf{N}\}$ is bounded (both above and below).

We say that a sequence is *unbounded* if it is not bounded.

Theorem 3.12. *A sequence (x_n) is bounded if and only if there exists $C > 0$ such that*

$$|x_n| \leqslant C \qquad \text{for all } n \in \mathbf{N}.$$

Proof. If (x_n) is bounded then there exist $\alpha, \beta \in \mathbf{R}$ such that

$$\alpha \leqslant x_n \leqslant \beta \qquad \text{for all } n \in \mathbf{N}.$$

Let $C = \max\{|\alpha|, |\beta|\}$. Then $-C \leqslant \alpha \leqslant \beta \leqslant C$, so that $-C \leqslant x_n \leqslant C$ for all $n \in \mathbf{N}$, so $|x_n| \leqslant C$ for all $n \in \mathbf{N}$.

Conversely, if $|x_n| \leqslant C$ for all $n \in \mathbf{N}$ then $-C \leqslant x_n \leqslant C$ for all $n \in \mathbf{N}$, so (x_n) is bounded. ☐

Theorem 3.13. *Let (x_n) be a sequence. If (x_n) converges, then (x_n) is bounded.*

Proof. Let L be the limit of (x_n).

Let $\varepsilon = 1$. There exists $M \in \mathbf{N}$ such that for all $n \in \mathbf{N}$ we have

$$n > M \Rightarrow |x_n - L| < 1 \Rightarrow L - 1 < x_n < L + 1.$$

The set $X = \{x_0, x_1, \ldots, x_M\}$ is finite, so we can take $\gamma = \min X$ and $\delta = \max X$. Let $\alpha = \min\{\gamma, L-1\}$ and $\beta = \max\{\delta, L+1\}$, then

$$\alpha \leqslant x_n \leqslant \beta \qquad \text{for all } n \in \mathbf{N},$$

in other words (x_n) is bounded. □

Taking the contrapositive we obtain:

Corollary 3.14. *Let (x_n) be a sequence. If (x_n) is unbounded, then (x_n) diverges.*

3.4 Properties of limits

Theorem 3.15 (Algebra of Limits)**.** *Let* (x_n) *and* (y_n) *be convergent sequences with* $x_n \longrightarrow \alpha$ *and* $y_n \longrightarrow \beta$*, where* $\alpha, \beta \in \mathbf{R}$*. Then*

(a) $x_n + y_n \longrightarrow \alpha + \beta$*;*

(b) $x_n - y_n \longrightarrow \alpha - \beta$*;*

(c) $x_n y_n \longrightarrow \alpha\beta$*;*

(d) $x_n / y_n \longrightarrow \alpha/\beta$*, if* $\beta \neq 0$ *and* $y_n \neq 0$ *for all* $n \in \mathbf{N}$*.*

Proof.

(a) We want to prove that

$$(\forall \varepsilon > 0)(\exists M \in \mathbf{N})(\forall n \in \mathbf{N}) n > M \Rightarrow |(x_n + y_n) - (\alpha + \beta)| < \varepsilon.$$

Let $\varepsilon > 0$.

Since $x_n \longrightarrow \alpha$, there exists $M_x \in \mathbf{N}$ such that for all $n \in \mathbf{N}$ we have

$$n > M_x \Rightarrow |x_n - \alpha| < \frac{\varepsilon}{2}.$$

Similarly, since $y_n \longrightarrow \alpha$, there exists $M_y \in \mathbf{N}$ such that for all $n \in \mathbf{N}$ we have

$$n > M_y \Rightarrow |y_n - \beta| < \frac{\varepsilon}{2}.$$

Let $M = \mathsf{max}\{M_x, M_y\}$. Let $n \in \mathbf{N}$. We have

$$\begin{aligned}
&n > M \Rightarrow n > M_x \Rightarrow |x_n - \alpha| < \frac{\varepsilon}{2} \\
&n > M \Rightarrow n > M_y \Rightarrow |y_n - \beta| < \frac{\varepsilon}{2} \\
&n > M \Rightarrow |(x_n + y_n) - (\alpha + \beta)| \leqslant |x_n - \alpha| + |y_n - \beta| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

(c) We want to prove that

$$(\forall \varepsilon > 0)(\exists M \in \mathbf{N})(\forall n \in \mathbf{N}) n > M \Rightarrow |x_n y_n - \alpha\beta| < \varepsilon.$$

Before we start in earnest: since (y_n) converges, it is bounded (by Theorem 3.13), so by Theorem 3.12 there exists $C > 0$ such that $|y_n| \leqslant C$ for all $n \in \mathbf{N}$.

Also, we will assume that $\alpha \neq 0$. The remaining case $(\alpha = 0)$ is Exercise 3.5.

Now let $\varepsilon > 0$. Since $x_n \longrightarrow \alpha$, there exists $M_x \in \mathbf{N}$ such that for all $n \in \mathbf{N}$ we have

$$n > M_x \Rightarrow |x_n - \alpha| < \frac{\varepsilon}{2C}.$$

Similarly, since $y_n \longrightarrow \alpha$, there exists $M_y \in \mathbf{N}$ such that for all $n \in \mathbf{N}$ we have (recall that $\alpha \neq 0$)

$$n > M_y \Rightarrow |y_n - \beta| < \frac{\varepsilon}{2|\alpha|}.$$

Let $M = \mathsf{max}\,\{M_x, M_y\}$. Let $n \in \mathbf{N}$. We have

$$\begin{aligned}
&n > M \Rightarrow n > M_x \Rightarrow |x_n - \alpha| < \frac{\varepsilon}{2C}\\
&n > M \Rightarrow n > M_y \Rightarrow |y_n - \beta| < \frac{\varepsilon}{2|\alpha|}\\
&n > M \Rightarrow |x_n y_n - \alpha\beta| = |(x_n - \alpha)y_n + \alpha(y_n - \beta)| \leqslant |x_n - \alpha||y_n| + |\alpha||y_n - \beta| \leqslant \frac{\varepsilon}{2C}C + |\alpha|\frac{\varepsilon}{2|\alpha|} = \varepsilon.
\end{aligned}$$

□

Theorem 3.16 (Inequalities and Limits)**.** *Let* (x_n) *and* (y_n) *be convergent sequences. If* $x_n \leqslant y_n$ *for all* $n \in \mathbf{N}$ *then*

$$\lim_{n\to\infty} x_n \leqslant \lim_{n\to\infty} y_n.$$

Proof. Let $x_n \longrightarrow \alpha$ and $y_n \longrightarrow \beta$, where $\alpha, \beta \in \mathbf{R}$. The claim is that $\alpha \leqslant \beta$.

We proceed by contradiction. Suppose $\alpha > \beta$. Let $\varepsilon = (\alpha - \beta)/3 > 0$. Then $\beta + \varepsilon < \alpha + \varepsilon$.

Since $x_n \longrightarrow \alpha$, there exists $M_x \in \mathbf{N}$ such that

$$(\forall n \in \mathbf{N}) n > M_x \Rightarrow |x_n - \alpha| < \varepsilon \Rightarrow \alpha - \varepsilon < x_n < \alpha + \varepsilon.$$

Similarly, since $y_n \longrightarrow \beta$, there exists $M_y \in \mathbf{N}$ such that

$$(\forall n \in \mathbf{N}) n > M_y \Rightarrow |y_n - \beta| < \varepsilon \Rightarrow \beta - \varepsilon < y_n < \beta + \varepsilon.$$

Let $M = \mathsf{max}\{M_x, M_y\}$. Let $n \in \mathbf{N}$. If $n > M$, then $n > M_x$ and $n > M_y$, therefore

$$\beta - \varepsilon < y_n < \beta + \varepsilon < \alpha - \varepsilon < x_n < \alpha + \varepsilon.$$

This implies that $x_n > y_n$, contradiction. $\square$

Corollary 3.17. *If* $a \in \mathbf{R}$ *is such that* $a \leqslant x_n$ *for all* $n \in \mathbf{N}$*, then*

$$a \leqslant \lim_{n\to\infty} x_n.$$

If $b \in \mathbf{R}$ *is such that* $x_n \leqslant b$ *for all* $n \in \mathbf{N}$*, then*

$$\lim_{n\to\infty} x_n \leqslant b.$$

Have a look at Tutorial Question 5.10.

3.5 Subsequences

Definition 3.18. Let (x_n) be a sequence. Let $n_0, n_1, n_2, \dots$ be integers such that

$$0 \leqslant n_0 < n_1 < n_2 < \dots.$$

We define

$$(x_{n_k})_k = (x_{n_0}, x_{n_1}, x_{n_2}, \dots)$$

and we call (x_{n_k}) a *subsequence* of (x_n).

Example 3.19. Consider $(x_n) = \big((-1)^n\big) = (1, -1, 1, -1, 1, -1, \dots)$.

Here are two subsequences of (x_n):

$$\big(x_{2k}\big) = (1, 1, 1, 1, \dots), \qquad \big(x_{2k+1}\big) = (-1, -1, -1, -1, \dots).$$

Lemma 3.20. *Let (x_n) be a sequence and $\big(x_{n_k}\big)$ a subsequence of (x_n). Then $n_k \geqslant k$ for all $k \in \mathbf{N}$.*

See Exercise 3.11.

Theorem 3.21. *Let $L \in \mathbf{R}$. A sequence (x_n) converges to L if and only if every subsequence of (x_n) converges to L.*

Proof. Suppose every subsequence of (x_n) converges to L. Since (x_n) itself is a subsequence of (x_n) (just take $n_k = k$ for all $k \in \mathbf{N}$), we get that (x_n) converges to L.

Conversely, suppose that (x_n) converges to L and let $\left(x_{n_k}\right)$ be a subsequence. By Lemma 3.20 we have $n_k \geqslant k$ for all $k \in \mathbf{N}$.

Let $\varepsilon > 0$. Since $x_n \longrightarrow L$, there exists $M \in \mathbf{N}$ such that for all $n \in \mathbf{N}$

$$n > M \Rightarrow |x_n - L| < \varepsilon.$$

Let $k \in \mathbf{N}$, since $n_k \geqslant k$ we have

$$k > M \Rightarrow n_k > M \Rightarrow |x_{n_k} - L| < \varepsilon.$$

Hence $\left(x_{n_k}\right)$ converges to L. ☐

Theorem 3.22 (Divergence Criteria). *Let (x_n) be a sequence. If any of the following statements is* True, *then (x_n) diverges:*

(a) (x_n) is unbounded;

(b) (x_n) has a subsequence that diverges;

(c) (x_n) has two subsequences that converge to different limits.

Proof.

(a) Contrapositive of Theorem 3.13.

(b) Contrapositive of Theorem 3.21.

(c) Contrapositive of Theorem 3.21. ☐

Example 3.23.

$$(x_n) = \big((-1)^n\big) = (1, -1, 1, -1, \dots)$$ is divergent because $\big(x_{2k}\big) \longrightarrow 1$ and $\big(x_{2k+1}\big) \longrightarrow -1$.

$$(x_n) = \big((-1)^n n\big) = (0, -1, 2, -3, 4, -5, \dots)$$ is divergent because it is unbounded.

3.6 Monotone sequences

Definition 3.24. Let (x_n) be a sequence. We say that (x_n) is

- *monotone increasing* if
$$x_n \leqslant x_{n+1} \qquad \text{for all } n \in \mathbf{N};$$
- *monotone decreasing* if
$$x_n \geqslant x_{n+1} \qquad \text{for all } n \in \mathbf{N};$$
- *monotone* if it is monotone increasing or monotone decreasing.

Example 3.25.

$(x_n) = \left(\frac{1}{n}\right)$ is monotone decreasing, hence monotone

$(x_n) = \left(1 - \frac{1}{n}\right)$ is monotone increasing, hence monotone

$(x_n) = (3, 3, 3, 3, 3, \dots)$ is both monotone increasing and monotone decreasing, hence monotone

$(x_n) = \left((-1)^n n\right)$ is neither monotone increasing nor monotone decreasing, hence not monotone.

Theorem 3.26. *Let (x_n) be a bounded sequence. If (x_n) is monotone, then (x_n) converges.*

Proof. We do the case where (x_n) is monotone increasing (the other is an exercise in flipping inequalities).

Suppose (x_n) is monotone increasing and bounded. Consider the set $X = \{x_n : \ n \in \mathbf{N}\}$, then X is bounded (in particular, bounded above). By the Completeness Axiom, $\sup X$ exists. Let $L = \sup X$. I claim that $x_n \longrightarrow L$.

Let $\varepsilon > 0$. Then $L - \varepsilon$ is not an upper bound of X, so there exists $M \in \mathbf{N}$ such that $x_M > L - \varepsilon$.

Suppose $n \in \mathbf{N}$ is such that $n > M$. Since (x_n) is increasing, $x_n \geqslant x_M > L - \varepsilon$. Therefore

$$n > M \Rightarrow L + \varepsilon > L \geqslant x_n > L - \varepsilon \Rightarrow |x_n - L| < \varepsilon.$$

□

Corollary 3.27. *Let (x_n) be a monotone sequence. Then (x_n) converges if and only if (x_n) is bounded.*

Proof. Suppose (x_n) converges, then (x_n) is bounded by Theorem 3.13.

In the other direction, suppose (x_n) is bounded. Since (x_n) is also monotone, then (x_n) converges by Theorem 3.26. ☐

Corollary 3.28. *Let (x_n) be a bounded sequence and let $X = \{x_n : n \in \mathbf{N}\}$.*

(a) If (x_n) is monotone increasing, then $x_n \longrightarrow \sup X$.

(b) If (x_n) is monotone decreasing, then $x_n \longrightarrow \inf X$.

This is simply stating a fact that we proved in the course of proving Theorem 3.26.

Definition 3.29. Let (x_n) be a sequence. We say that $p \in \mathbf{N}$ is a *peak* of (x_n) if

$$x_n \leqslant x_p \qquad \text{for all } n > p.$$

A picture?

Example 3.30. The sequence

$$(x_n) = \left(2, 1, 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\right)$$

has only two peaks: $n = 0$ and $n = 1$.

Example 3.31. The sequence

$$(x_n)_{n \geqslant 1} = \left(\frac{(-1)^n}{n}\right)_{n \geqslant 1} = \left(-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \ldots\right)$$

has infinitely many peaks: for each $k \in \mathbf{N}$, $2k$ is a peak.

Lemma 3.32. *Let (x_n) be a bounded sequence.*

(a) If (x_n) has finitely many peaks, then (x_n) has a monotone increasing subsequence.

(b) If (x_n) has infinitely many peaks, then (x_n) has a monotone decreasing subsequence.

Proof.

(a) Suppose (x_n) has finitely many peaks. Let p be the (index of the) largest peak, then (x_n) does not have a peak at n for all $n > p$.

Let $n_0 = p + 1$. Since (x_n) does not have a peak at n_0, there exists $n_1 > n_0$ such that $x_{n_0} < x_{n_1}$. Since (x_n) does not have a peak at n_1, there exists $n_2 > n_1$ such that $x_{n_1} < x_{n_2}$.

Continuing in this manner, we get $n_0 < n_1 < n_2 < \dots$ such that $x_{n_k} < x_{n_{k+1}}$ for all $k \in \mathbf{N}$. Therefore $\left(x_{n_k}\right)$ is monotone increasing.

(b) Suppose (x_n) has infinitely many peaks. Enumerate these peaks in increasing order: $n_0 < n_1 < n_2 < \dots$.

For every $k \in \mathbf{N}$, since (x_n) has a peak at n_k, we have $x_{n_{k+1}} \leqslant x_{n_k}$. Therefore $\left(x_{n_k}\right)$ is monotone decreasing. □

Theorem 3.33 (Bolzano–Weierstrass)**.** *Let* (x_n) *be a sequence. If* (x_n) *is bounded, then it has a convergent subsequence.*

Proof. Follows directly from Lemma 3.32: either (x_n) has finitely many peaks or it has infinitely many peaks. In either case, we know that (x_n) has a monotone subsequence (x_{n_k}), which is also bounded (since (x_n) is bounded). By Theorem 3.26, (x_{n_k}) converges. □

3.7 Cauchy sequences and completeness

If (x_n) is a sequence that converges to $L \in \mathbf{R}$, then as $n \longrightarrow \infty$, the terms x_n are getting closer and closer to L.

It follows that they are also getting closer and closer to each other.

Definition 3.34. Let (x_n) be a sequence. We say that (x_n) is a *Cauchy sequence* if for every $\varepsilon > 0$ there exists $M \in \mathbf{N}$ such that

$$(\forall j, k \in \mathbf{N}) j, k > M \Rightarrow |x_j - x_k| < \varepsilon.$$

Example 3.35. Consider the sequence

$$(x_n) = \left(\frac{(-1)^n}{n}\right).$$

I claim that (x_n) is Cauchy.

Let $\varepsilon > 0$. Define $M = \lceil 2/\varepsilon \rceil$. If $j, k \in \mathbf{N}$ satisfy $j, k > M$, then by the triangle inequality

$$|x_j - x_k| = \left|\frac{(-1)^j}{j} - \frac{(-1)^k}{k}\right| \leqslant \left|\frac{(-1)^j}{j}\right| + \left|\frac{(-1)^k}{k}\right| = \frac{1}{j} + \frac{1}{k} < \frac{1}{M} + \frac{1}{M} = \frac{2}{M} \leqslant \varepsilon.$$

Theorem 3.36. *Let* (x_n) *be a sequence. If* (x_n) *is Cauchy, then* (x_n) *is bounded.*

Proof. Let $\varepsilon = 1$. There exists $M \in \mathbf{N}$ such that for all $j, k \in \mathbf{N}$

$$j, k > M \Rightarrow |x_j - x_k| < 1.$$

In particular, if $k > M$ then $|x_k| < |x_{M+1}| + 1$.

Let $C = \max\{|x_0|, |x_1|, \ldots, |x_M|, |x_{M+1}| + 1\}$, then $|x_k| \leqslant C$ for all $k \in \mathbf{N}$. $\square$

Theorem 3.37. *Let* (x_n) *be a Cauchy sequence and* $\left(x_{n_k}\right)$ *a subsequence that converges to* $L \in \mathbf{R}$*. Then* $x_n \longrightarrow L$*.*

This is of course not True if (x_n) is not a Cauchy sequence!

Proof. We proceed by contradiction. Suppose (x_n) does not converge to L. This means that there exists $\varepsilon > 0$ such that:

$$(\forall M \in \mathbf{N})(\exists n \in \mathbf{N}) n > M \text{ and } |x_n - L| \geqslant \varepsilon. \qquad (*).$$

Fix such $\varepsilon > 0$.

Since (x_n) is Cauchy there exists $M_1 \in \mathbf{N}$ such that for all $i, j \in \mathbf{N}$ we have

$$i, j > M_1 \Rightarrow |x_i - x_j| < \frac{\varepsilon}{2}.$$

Since $x_{n_k} \longrightarrow L$ there exists $M_2 \in \mathbf{N}$ such that for all $k \in \mathbf{N}$ we have

$$k > M_2 \Rightarrow |x_{n_k} - L| < \frac{\varepsilon}{2}.$$

Let $M = \max\{M_1, M_2\}$. Take an arbitrary $n > M$. Let $k > M$, then $n_k > M \geqslant M_2$, so

$$|x_{n_k} - L| < \frac{\varepsilon}{2}.$$

On the other hand, we have $n, n_k > M \geqslant M_1$ so

$$|x_n - x_{n_k}| < \frac{\varepsilon}{2}.$$

Combining the last two inequalities we have

$$|x_n - L| \leqslant |x_n - x_{n_k}| + |x_{n_k} - L| < \varepsilon,$$

which contradicts (*). □

Theorem 3.38. *Let (x_n) be a sequence. Then (x_n) converges if and only if it is Cauchy.*

Note: this is what is more generally known as "completeness".

Proof. Suppose (x_n) converges and let $L \in \mathbf{R}$ be its limit.

Let $\varepsilon > 0$. There exists $M \in \mathbf{N}$ such that for all $n \in \mathbf{N}$

$$n > M \Rightarrow |x_n - L| < \frac{\varepsilon}{2}.$$

Therefore, for all $j, k \in \mathbf{N}$

$$j, k > M \Rightarrow |x_j - x_k| = |(x_j - L) - (x_k - L)| \leqslant |x_j - L| + |x_k - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

We conclude that (x_n) is Cauchy.

Conversely, suppose (x_n) is Cauchy. By Theorem 3.36 we know that (x_n) is bounded. By the Bolzano–Weierstrass Theorem we know that (x_n) has a convergent subsequence. By Theorem 3.37, (x_n) converges. ☐

We illustrate the power of the Cauchy convergence criterion with an application.

Definition 3.39. Let (x_n) be a sequence. We say that (x_n) is *contractive* if there exists $0 < c < 1$ such that

$$|x_{n+2} - x_{n+1}| \leqslant c|x_{n+1} - x_n| \qquad \text{for all } n \in \mathbf{N}.$$

This is a fairly strong condition on the sequence. We will soon see that it implies convergence.

Example 3.40. Define a sequence (x_n) by letting $x_0 = \frac{1}{4}$ and

$$x_{n+1} = \frac{x_n^7 + 1}{5} \qquad \text{for all } n \in \mathbf{N}.$$

I claim that $0 < x_n < 1/2$ for all $n \in \mathbf{N}$.

We proceed by induction.

Base case $n = 0$: $x_0 = 1/4$ so True.

Induction step: let $k \geqslant 0$ be arbitrary but fixed and suppose that $0 < x_k < 1/2$. We have

$$\begin{aligned} 0 <& x_k^7 < \frac{1}{128} \\ 1 <& x_k^7 + 1 < \frac{129}{128} \\ 0 < \frac{1}{5} <& \frac{x_k^7 + 1}{5} < \frac{129}{640} < \frac{1}{2}. \end{aligned}$$

So $0 < x_{k+1} < 1/2$.

Now we show that (x_n) is contractive.

Let $n \geqslant 0$. We have

$$\begin{aligned} |x_{n+2} - x_{n+1}| &= \left| \frac{x_{n+1}^7 + 1}{5} - \frac{x_n^7 + 1}{5} \right| = \frac{1}{5}|x_{n+1}^7 - x_n^7| \\ &= \frac{1}{5}|x_{n+1} - x_n||x_{n+1}^6 + x_{n+1}^5 x_n + \cdots + x_{n+1} x_n^5 + x_n^6| \\ &\leqslant \frac{1}{5}|x_{n+1} - x_n|\big(|x_{n+1}|^6 + |x_{n+1}|^5|x_n| + \cdots + |x_{n+1}||x_n|^5 + |x_n|^6\big) \\ &\leqslant \frac{1}{5}|x_{n+1} - x_n| 7 \left(\frac{1}{2}\right)^6 = \frac{7}{320}|x_{n+1} - x_n|. \end{aligned}$$

So (x_n) is contractive with $x = 7/320$.

Lemma 3.41. *Let (x_n) be a contractive sequence with constant c. Then*

$$|x_{n+1} - x_n| \leqslant c^n \, |x_1 - x_0| \qquad \text{for all } n \in \mathbf{N}.$$

In particular, if $x_1 = x_0$ then the sequence is constant $(x_n) = (x_0, x_0, x_0, \dots)$.

See **??**.

Theorem 3.42. *Let (x_n) be a sequence. If (x_n) is contractive, then it converges.*

Proof. By the Lemma, if $x_1 = x_0$ then (x_n) is constant, so it clearly converges.

It remains to deal with the case where $x_1 \neq x_0$. Again by the Lemma, we have that for all $n \in \mathbf{N}$:

$$|x_{n+1} - x_n| \leqslant c^n \, |x_1 - x_0|.$$

We will prove that (x_n) is a Cauchy sequence.

Fix $M \in \mathbf{N}$ and suppose m, n are such that $n > m > M$. Then

$$\begin{aligned}
|x_n - x_m| &= |(x_n - x_{n-1}) + (x_{n-1} - x_{n-2}) + \cdots + (x_{m+2} - x_{m+1}) + (x_{m+1} - x_m)| \\
&\leqslant |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \cdots + |x_{m+2} - x_{m+1}| + |x_{m+1} - x_m| \\
&\leqslant c^{n-1}|x_1 - x_0| + c^{n-2}|x_1 - x_0| + \cdots + c^{m+1}|x_1 - x_0| + c^m|x_1 - x_0| \\
&= c^m \left(c^{n-m-1} + c^{n-m-2} + \cdots + c + 1\right) |x_1 - x_0| \\
&= c^m \frac{1 - c^{n-m}}{1 - c}|x_1 - x_0| \leqslant (c^m)\frac{|x_1 - x_0|}{1 - c} \leqslant (c^M)\frac{|x_1 - x_0|}{1 - c}.
\end{aligned}$$

Let $\varepsilon > 0$. Since $0 < c < 1$, $c^n \longrightarrow 0$ as $n \longrightarrow \infty$, so there exists $N \in \mathbf{N}$ such that for all $j > N$ we have

$$c^j < \varepsilon \frac{1 - c}{|x_1 - x_0|}.$$

Take $M = N + 1$ and consider n, m with $n > m > M$, then by the above inequalities we have

$$|x_n - x_m| \leqslant (c^M)\frac{|x_1 - x_0|}{1 - c} < \varepsilon.$$

We conclude that (x_n) is Cauchy, therefore convergent. □

Example 3.43. In particular, the sequence (x_n) from Example 3.40 converges. What is the limit?

Let L be the limit.

Recall that

$$x_{n+1} = \frac{x_n^7 + 1}{5} \qquad \text{for all } n \in \mathbf{N}.$$

Taking the limit as $n \to \infty$ on both sides of the equality and using the Algebra of Limits we have

$$L = \frac{L^7 + 1}{5} \qquad \Rightarrow \qquad L^7 - 5L + 1 = 0.$$

In addition, as $0 < x_n < 1/2$ for all $n \in \mathbf{N}$, we know that $0 \leqslant L \leqslant 1/2$.

So L is a real root of the polynomial function $f(x) = x^7 - 5x + 1$ located in the interval $[0, 1/2]$.

We can think of (x_n) as a sequence of increasingly accurate approximations to the root L. This sequence converges very fast to L; already when $n = 5$ we have

$$L \approx x_5 = 0.200002560229405\ldots,$$

where all the displayed decimals are correct.

Our focus has so far been mainly on convergent sequences. There is a special case of divergence that is very useful and worth pointing out.

Definition 3.44. Let (x_n) be a sequence.

- We say that (x_n) *diverges to* ∞, and write $x_n \longrightarrow \infty$, if for every $C \in \mathbf{R}$ there exists $M \in \mathbf{N}$ such that for all $n \in \mathbf{N}$ we have
$$n > M \Rightarrow x_n > C.$$
- We say that (x_n) *diverges to* $-\infty$, and write $x_n \longrightarrow -\infty$, if for every $C \in \mathbf{R}$ there exists $M \in \mathbf{N}$ such that for all $n \in \mathbf{N}$ we have
$$n > M \Rightarrow x_n < C.$$

Example 3.45.

- The sequence $(x_n) = (n) = (0, 1, 2, 3, 4, \dots)$ diverges to ∞.
- The sequence $(x_n) = (-n) = (0, -1, -2, -3, -4, \dots)$ diverges to $-\infty$.
- The sequence $(x_n) = \big((-1)^n n\big) = (0, -1, 2, -3, 4, -5, \dots)$ diverges, but does not diverge to ∞ or to $-\infty$.