3 Sequences

Chapter contents

3.1 Definition and examples

Recall that, in our convention, the first natural number is 0:

$$\mathbf{N} = \{0, 1, 2, 3, \ldots\}.$$

Definition 3.1. A *sequence* (of real numbers) is a function $x : \mathbf{N} \longrightarrow \mathbf{R}$.

We usually denote $x(n)$ by x_n and we call it the n-th *term* of the sequence; we denote the sequence itself by $(x_n) = (x_0, x_1, x_2, x_3, \ldots)$.

Example 3.2. $(x_n) = (n^2 + 1)$

Sometimes we want to start the sequence at a point $n_0 \in \mathbf{N}$, $n_0 > 0$. In this case we would denote the sequence as

$$\big(x_n\big)_{n \geqslant n_0} = \big(x_{n_0}, x_{n_0+1}, x_{n_0+2}, \ldots\big).$$

Example 3.3.

$$\big(x_n\big)_{n \geqslant 1} = \left(\frac{1}{n}\right)_{n \geqslant 1}$$

Occasionally people write simply $(1/n)$ for this sequence, and we have to infer that the starting point should be 1 instead of 0.

Example 3.4. We can "approximate" π by the sequence

$$(x_n) = (3, 3.1, 3.14, 3.141, 3.1416, \dots).$$

Intuitively, the terms of this sequence converge to the real number π.

On the other hand, the sequence $\big((-1)^n\big)$

Neither does the sequence $(2n)$

Example 3.5. Consider the sequence

$$(x_n)_{n\geqslant 1} = \left(4 + (-1)^n \frac{2}{\sqrt{n}}\right)_{n\geqslant 1} = \left(4 - \frac{2}{\sqrt{1}}, 4 + \frac{2}{\sqrt{2}}, 4 - \frac{2}{\sqrt{3}}, 4 + \frac{2}{\sqrt{4}}, \dots\right)$$
$$\approx \big(2, 5.41, 2.85, 5, 3.11, 4.82, 3.24, 4.71, \dots\big).$$

Not so clear what's going on? Computing further:

n	x_n	n	x_n
100		999	
10000		99999	
1000000		9999999	

We are trying to express mathematically the situation: "as n gets larger and larger, the term x_n gets closer and closer to 4".

Let's note that this is the same as: "as n gets larger and larger, the distance $|x_n - 4|$ gets smaller and smaller".

3.2 Limit and convergence

Definition 3.6. Let (x_n) be a sequence and let $L \in \mathbf{R}$.

We say that (x_n) *converges to L* if: for every $\varepsilon > 0$ there exists $M \in \mathbf{N}$ such that for all $n \in \mathbf{N}$, if $n > M$ then $|x_n - L| < \varepsilon$.

In this case, we write

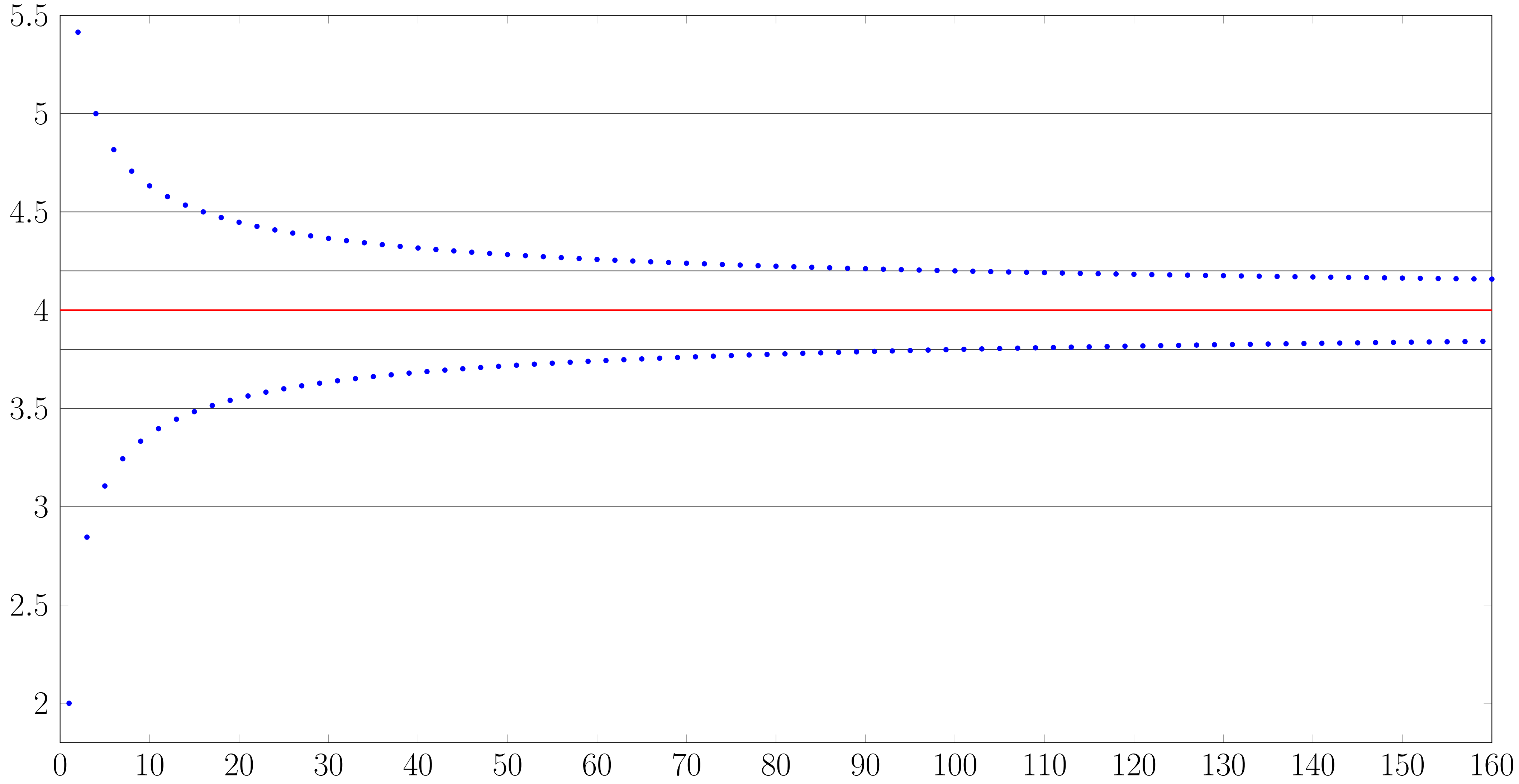
$$x_n \longrightarrow L \qquad \text{or} \qquad \lim_{n\to\infty} x_n = L$$

and call L a *limit* of the sequence (x_n).

In formal logic notation, the definition of "(x_n) converges to L" is

Its negation "(x_n) does not converge to L" is

Going back to the sequence from Example 3.5:



Example 3.7. Prove that $1 - \dfrac{1}{n} \longrightarrow 1$.

Scrap work. Fix $\varepsilon > 0$ and "solve for" M:

Proof.

We return to:

Example 3.8. Prove that

$$\left(4 + (-1)^n \frac{2}{\sqrt{n}}\right) \longrightarrow 4.$$

Scrap work. Fix $\varepsilon > 0$ and "solve for" M:

Proof.

Definition 3.9. Let (x_n) be a sequence.

We say that (x_n) *converges* (without reference to a particular limit) if there exists $L \in \mathbf{R}$ such that $x_n \longrightarrow L$.

We say that (x_n) *diverges* if it does not converge.

So (x_n) diverges if for all $L \in \mathbf{R}$, the statement "(x_n) converges to L" is False. In formal logic notation:

Example 3.10. Consider the sequence $\big((-1)^n\big)$

3.3 Bounded sequences

The sequence $(n^2) = (0, 1, 4, 9, 16, \dots)$ also looks like it should diverge, but for a different reason.

Definition 3.11. We say that a sequence (x_n) is *bounded* if the set $\{x_n : \ n \in \mathbf{N}\}$ is bounded (both above and below).

We say that a sequence is *unbounded* if it is not bounded.

Theorem 3.12. *A sequence (x_n) is bounded if and only if there exists $C > 0$ such that*

$$|x_n| \leqslant C \qquad \textit{for all } n \in \mathbf{N}.$$

Theorem 3.13. *Let* (x_n) *be a sequence. If* (x_n) *converges, then* (x_n) *is bounded.*

Taking the contrapositive we obtain:

Corollary 3.14. *Let* (x_n) *be a sequence. If* (x_n) *is unbounded, then* (x_n) *diverges.*

3.4 Properties of limits

Theorem 3.15 (Algebra of Limits)**.** *Let* (x_n) *and* (y_n) *be convergent sequences with* $x_n \longrightarrow \alpha$ *and* $y_n \longrightarrow \beta$*, where* $\alpha, \beta \in \mathbf{R}$*. Then*

(a) $x_n + y_n \longrightarrow \alpha + \beta$*;*

(b) $x_n - y_n \longrightarrow \alpha - \beta$*;*

(c) $x_n y_n \longrightarrow \alpha\beta$*;*

(d) $x_n / y_n \longrightarrow \alpha/\beta$*, if* $\beta \neq 0$ *and* $y_n \neq 0$ *for all* $n \in \mathbf{N}$*.*

Theorem 3.16 (Inequalities and Limits)**.** *Let* (x_n) *and* (y_n) *be convergent sequences. If* $x_n \leqslant y_n$ *for all* $n \in \mathbf{N}$ *then*

$$\lim_{n\to\infty} x_n \leqslant \lim_{n\to\infty} y_n.$$

Corollary 3.17. *If* $a \in \mathbf{R}$ *is such that* $a \leqslant x_n$ *for all* $n \in \mathbf{N}$*, then*

$$a \leqslant \lim_{n\to\infty} x_n.$$

If $b \in \mathbf{R}$ *is such that* $x_n \leqslant b$ *for all* $n \in \mathbf{N}$*, then*

$$\lim_{n\to\infty} x_n \leqslant b.$$

3.5 Subsequences

Definition 3.18. Let (x_n) be a sequence. Let $n_0, n_1, n_2, \dots$ be integers such that

$$0 \leqslant n_0 < n_1 < n_2 < \dots .$$

We define

$$\left(x_{n_k}\right)_k = \left(x_{n_0}, x_{n_1}, x_{n_2}, \dots\right)$$

and we call (x_{n_k}) a *subsequence* of (x_n).

Example 3.19. Consider $(x_n) = \left((-1)^n\right) = (1, -1, 1, -1, 1, -1, \dots)$.

Lemma 3.20. *Let* (x_n) *be a sequence and* $\left(x_{n_k}\right)$ *a subsequence of* (x_n)*. Then* $n_k \geqslant k$ *for all* $k \in \mathbf{N}$*.*

Theorem 3.21. *Let $L \in \mathbf{R}$. A sequence (x_n) converges to L if and only if every subsequence of (x_n) converges to L.*

Theorem 3.22 (Divergence Criteria)**.** *Let* (x_n) *be a sequence. If any of the following statements is* True, *then* (x_n) *diverges:*

- *(a)* (x_n) *is unbounded;*
- *(b)* (x_n) *has a subsequence that diverges;*
- *(c)* (x_n) *has two subsequences that converge to different limits.*

Example 3.23.

$$(x_n) = \big((-1)^n\big) = (1, -1, 1, -1, \dots)$$

$$(x_n) = \big((-1)^n n\big) = (0, -1, 2, -3, 4, -5, \dots)$$

3.6 Monotone sequences

Definition 3.24. Let (x_n) be a sequence. We say that (x_n) is

- *monotone increasing* if
$$x_n \leqslant x_{n+1} \qquad \text{for all } n \in \mathbf{N};$$
- *monotone decreasing* if
$$x_n \geqslant x_{n+1} \qquad \text{for all } n \in \mathbf{N};$$
- *monotone* if it is monotone increasing or monotone decreasing.

Example 3.25.

$$(x_n) = \left(\frac{1}{n}\right)$$

$$(x_n) = \left(1 - \frac{1}{n}\right)$$

$$(x_n) = (3, 3, 3, 3, 3, \dots)$$

$$(x_n) = \left((-1)^n n\right)$$

Theorem 3.26. *Let* (x_n) *be a bounded sequence. If* (x_n) *is monotone, then* (x_n) *converges.*

Corollary 3.27. *Let (x_n) be a monotone sequence. Then (x_n) converges if and only if (x_n) is bounded.*

Corollary 3.28. *Let (x_n) be a bounded sequence and let $X = \{x_n : \ n \in \mathbf{N}\}$.*

- *(a) If (x_n) is monotone increasing, then $x_n \longrightarrow \sup X$.*
- *(b) If (x_n) is monotone decreasing, then $x_n \longrightarrow \inf X$.*

Definition 3.29. Let (x_n) be a sequence. We say that $p \in \mathbf{N}$ is a *peak* of (x_n) if

$$x_n \leqslant x_p \qquad \text{for all } n > p.$$

Example 3.30. The sequence

$$(x_n) = \left(2, 1, 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots\right)$$

Example 3.31. The sequence

$$(x_n)_{n \geqslant 1} = \left(\frac{(-1)^n}{n}\right)_{n \geqslant 1}$$

Lemma 3.32. *Let* (x_n) *be a bounded sequence.*

(a) If (x_n) *has finitely many peaks, then* (x_n) *has a monotone increasing subsequence.*

(b) If (x_n) *has infinitely many peaks, then* (x_n) *has a monotone decreasing subsequence.*

Theorem 3.33 (Bolzano–Weierstrass)**.** *Let* (x_n) *be a sequence. If* (x_n) *is bounded, then it has a convergent subsequence.*

3.7 Cauchy sequences and completeness

If (x_n) is a sequence that converges to $L \in \mathbf{R}$, then as $n \longrightarrow \infty$, the terms x_n are getting closer and closer to L.

It follows that

Definition 3.34. Let (x_n) be a sequence. We say that (x_n) is a *Cauchy sequence* if for every $\varepsilon > 0$ there exists $M \in \mathbf{N}$ such that

$$(\forall j, k \in \mathbf{N}) j, k > M \Rightarrow |x_j - x_k| < \varepsilon.$$

Example 3.35. Consider the sequence

$$(x_n) = \left(\frac{(-1)^n}{n} \right).$$

Theorem 3.36. *Let (x_n) be a sequence. If (x_n) is Cauchy, then (x_n) is bounded.*

Theorem 3.37. *Let* (x_n) *be a Cauchy sequence and* $\left(x_{n_k}\right)$ *a subsequence that converges to* $L \in \mathbf{R}$*. Then* $x_n \longrightarrow L$*.*

This is of course not True if (x_n) is not a Cauchy sequence!

Theorem 3.38. *Let* (x_n) *be a sequence. Then* (x_n) *converges if and only if it is Cauchy.*

We illustrate the power of the Cauchy convergence criterion with an application.

Definition 3.39. Let (x_n) be a sequence. We say that (x_n) is *contractive* if there exists $0 < c < 1$ such that

$$|x_{n+2} - x_{n+1}| \leqslant c|x_{n+1} - x_n| \qquad \text{for all } n \in \mathbf{N}.$$

Example 3.40. Define a sequence (x_n) by letting $x_0 = \frac{1}{4}$ and

$$x_{n+1} = \frac{x_n^7 + 1}{5} \qquad \text{for all } n \in \mathbf{N}.$$

I claim that $0 < x_n < 1/2$ for all $n \in \mathbf{N}$.

Now we show that (x_n) is contractive.

Lemma 3.41. *Let* (x_n) *be a contractive sequence with constant* c*. Then*

$$|x_{n+1} - x_n| \leqslant c^n \, |x_1 - x_0| \qquad \text{for all } n \in \mathbf{N}.$$

In particular, if $x_1 = x_0$ *then the sequence is constant* $(x_n) = (x_0, x_0, x_0, \dots)$*.*

Theorem 3.42. *Let* (x_n) *be a sequence. If* (x_n) *is contractive, then it converges.*

Example 3.43. In particular, the sequence (x_n) from Example 3.40 converges. What is the limit?

Our focus has so far been mainly on convergent sequences. There is a special case of divergence that is very useful and worth pointing out.

Definition 3.44. Let (x_n) be a sequence.

- We say that (x_n) *diverges to* ∞, and write $x_n \longrightarrow \infty$, if for every $C \in \mathbf{R}$ there exists $M \in \mathbf{N}$ such that for all $n \in \mathbf{N}$ we have

$$n > M \Rightarrow x_n > C.$$

- We say that (x_n) *diverges to* $-\infty$, and write $x_n \longrightarrow -\infty$, if for every $C \in \mathbf{R}$ there exists $M \in \mathbf{N}$ such that for all $n \in \mathbf{N}$ we have

$$n > M \Rightarrow x_n < C.$$

Example 3.45.

- The sequence $(x_n) = (n) = (0, 1, 2, 3, 4, \dots)$
- The sequence $(x_n) = (-n) = (0, -1, -2, -3, -4, \dots)$
- The sequence $(x_n) = \big((-1)^n n\big) = (0, -1, 2, -3, 4, -5, \dots)$