6 Series

Chapter contents

6.1 Series of numbers

Definition 6.1. Let (a_n) be a sequence of real numbers. Let (A_n) be the sequence with terms

$$A_n = \sum_{k=0}^{n} a_k = a_0 + a_1 + \cdots + a_n.$$

The expression $\sum_{k=0}^{\infty} a_k$ is called the *series* with *partial sums* A_n.

Example 6.2.

$$\sum_{k=0}^{\infty} k, \qquad \sum_{k=1}^{\infty} \frac{1}{k^2}, \qquad \sum_{k=0}^{\infty} \frac{1}{4^k}, \qquad \ldots$$

Definition 6.3. Let L be a real number.

The series $\sum_{k=0}^{\infty} a_k$ *converges to L* if $\lim_{n\to\infty} A_n = L$.

In this case we write $\sum_{k=0}^{\infty} a_k = L$.

We say that the series *converges* if there exists $L \in \mathbf{R}$ such that the series converges to L.

Definition 6.4. We say that the series $\sum_{k=0}^{\infty} a_k$

- *diverges* if the sequence of partial sums (A_n) diverges;
- *diverges to ∞* if $\lim_{n\to\infty} A_n = \infty$;
- *diverges to $-\infty$* if $\lim_{n\to\infty} A_n = -\infty$.

Example 6.5.

$$\sum_{k=0}^{\infty} k = 0 + 1 + 2 + 3 + \dots$$

Theorem 6.6. *If the series* $\sum_{k=0}^{\infty} a_k$ *converges, then* $a_k \longrightarrow 0$.

The contrapositive of Theorem 6.6 gives us a divergence criterion:

Corollary 6.7. *If the sequence* (a_n) *does not converge to* 0*, then the series* $\sum_{n=0}^{\infty} a_n$ *diverges.*

Example 6.8.

$$\sum_{n=0}^{\infty} \frac{2n^2}{n^2+1}$$

Alternating series

Definition 6.9. An *alternating series* is a series of the form

$$\sum_{n=0}^{\infty} (-1)^n b_n = b_0 - b_1 + b_2 - b_3 + \dots,$$

where the sequence (b_n) has $b_n \geqslant 0$ for all n.

Theorem 6.10 (Alternating Series Test)**.** *Let* (b_n) *be a monotone decreasing sequence.*

The alternating series $\sum_{n=0}^{\infty} (-1)^n b_n$ *converges if and only if* $b_n \longrightarrow 0$.

Example 6.11.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

Definition 6.12. Given $r \in \mathbf{R}$, the *geometric series with ratio r* is the series

$$\sum_{k=0}^{\infty} r^k = 1 + r + r^2 + r^3 + \dots$$

Theorem 6.13 (Geometric Series Test)**.** *The geometric series with ratio r converges if and only if $|r| < 1$.*

When it converges, the limit is $1/(1-r)$.

Theorem 6.14 (Algebra of Series)**.** *If $\alpha, \beta, \lambda \in \mathbf{R}$ are such that*

$$\sum_{n=0}^{\infty} a_n = \alpha \qquad \textit{and} \qquad \sum_{n=0}^{\infty} b_n = \beta,$$

then

$$\sum_{n=0}^{\infty} (a_n + b_n) = \alpha + \beta \qquad \textit{and} \qquad \sum_{n=0}^{\infty} \lambda a_n = \lambda\alpha.$$

Absolute convergence

Definition 6.15. We say that the series $\sum_{k=0}^{\infty} a_k$ *converges absolutely* if the series $\sum_{k=0}^{\infty} |a_k|$ converges.

Theorem 6.16. *If a series converges absolutely, then any rearrangement of the series converges to the same value.*

(In particular, any absolutely convergent series converges.)

Positive series

Theorem 6.17. *Let (a_k) be a sequence with $a_k \geqslant 0$ for all k.*

The series $\sum_{k=0}^{\infty} a_k$ converges if and only if the sequence (A_n) of partial sums is bounded above.

Integral Test and p-series

Recall the notion of improper integral on an unbounded interval from Tutorial 11A.

Theorem 6.18 (Integral Test)**.** *Let* $f:\ [1,\infty) \longrightarrow \mathbf{R}$ *be positive, decreasing, and integrable on* $[1,b]$ *for all* $b>1$.

The series $\displaystyle\sum_{k=1}^{\infty} f(k)$ *converges if and only if the improper integral* $\displaystyle\int_1^{\infty} f(u)\,du$ *converges.*

Corollary 6.19 (p-series Test)**.** *Let* $p \in \mathbf{R}$, $p \geqslant 0$. *The series* $\sum_{k=1}^{\infty} \frac{1}{k^p}$

- *converges for all* $p > 1$;
- *diverges for all* p *such that* $0 \leqslant p \leqslant 1$.

Example 6.20. Famous special cases include

- the *harmonic series* $\sum_{k=1}^{\infty} \frac{1}{k}$
- $\sum_{k=1}^{\infty} \frac{1}{k^2}$
- $\sum_{k=1}^{\infty} \frac{1}{k^3}$

Ratio Test

Recall from Theorem 6.13 that a geometric series converges if and only if the ratio of consecutive terms $r = \frac{a_{k+1}}{a_k}$ satisfies $|r| < 1$.

We can generalise this to:

Theorem 6.21 (Ratio Test)**.** *Let (a_k) be a sequence and $r \in \mathbf{R}_{\geqslant 0}$ such that*

$$\lim_{k\to\infty} \left| \frac{a_{k+1}}{a_k} \right| = r.$$

Consider the series $\sum_{k=0}^{\infty} a_k$.

- *If $r < 1$, then the series converges.*
- *If $r > 1$, then the series diverges.*

Example 6.22.

$$\sum_{k=0}^{\infty} \frac{k^2}{2^k}$$

The Ratio Test says nothing about the case $r = 1$:

Example 6.23.

$$\sum_{n=1}^{\infty} \frac{1}{n} \qquad\qquad \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Conditional convergence

Examples 6.11 and 6.20 tell us that the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$

is convergent, but not absolutely convergent.

Definition 6.24. A series is *conditionally convergent* if it is convergent, but not absolutely convergent.

Theorem 6.25. *If a series is conditionally convergent, then for every $L \in \mathbf{R}$ there exists a rearrangement of the series that converges to L.*