6.2 Series of functions

We shift our attention from series of real numbers $\sum_{k=0}^{\infty} a_k$ to series of functions

$$\sum_{k=0}^{\infty} f_k(x).$$

The approach is similar: convergence of the series is defined by the convergence of the sequence (F_n) of partial sums

$$F_n(x) = \sum_{k=0}^{n} f_k(x).$$

The convergence can be pointwise or uniform.

Proving uniform convergence

We start with a uniform version of the Cauchy Convergence Criterion for sequences

Theorem 6.26 (Cauchy Uniform Convergence Criterion for Sequences)**.** *Let* $E \subseteq \mathbf{R}$ *and let* (f_n) *be a sequence of functions* $f_n : E \longrightarrow \mathbf{R}$*. This sequence converges uniformly on* E *if and only if for every* $\varepsilon > 0$ *there exists* $M \in \mathbf{N}$ *such that for all* $m, n > M$ *we have*

$$|f_m(x) - f_n(x)| < \varepsilon \qquad \text{for all } x \in E.$$

Corollary 6.27 (Cauchy Uniform Convergence Criterion for Series)**.** *Let* $E \subseteq \mathbf{R}$ *and consider the series* $\sum_{k=0}^{\infty} f_k$*, where* $f_k : E \longrightarrow \mathbf{R}$ *for all* $k \in \mathbf{N}$*. The series converges uniformly on* E *if and only if for every* $\varepsilon > 0$ *there exists* $M \in \mathbf{N}$ *such that for all* $m, n \in \mathbf{N}$*, if* $n > m > M$ *then*

$$|f_{m+1}(x) + \cdots + f_n(x)| < \varepsilon \qquad \text{for all } x \in E.$$

This leads to a very useful criterion:

Corollary 6.28 (Weierstrass M-test)**.** *Let* (M_k) *be a sequence of positive real numbers such that for all* $k \in \mathbf{N}$ *we have*

$$|f_k(x)| \leqslant M_k \qquad \text{for all } x \in E.$$

If the series $\sum_{k=0}^{\infty} M_k$ *converges, then the series* $\sum_{k=0}^{\infty} f_k$ *converges uniformly on* E*.*

Sequences of differentiable and integrable functions

Let (f_n) be a sequence of differentiable functions $f_n: [a,b] \longrightarrow \mathbf{R}$.

Based on our (limited) experience with sequences of continuous functions, we would perhaps expect that (a) if f_n converges **pointwise** to f, then f is not necessarily differentiable, but (b) if f_n converges **uniformly** to f, then f is differentiable.

However, even this last statement is too optimistic.

Example 6.29 (Weierstrass)**.** Consider $F_n(x) = \sum_{k=0}^{n} \frac{\cos(3^k x)}{2^k}$.

In order to guarantee that the limit is differentiable, we need to assume something about the sequence (f_n') of derivatives.

Theorem 6.30. *For each* $n \in \mathbf{N}$*, let* $f_n : [a, b] \longrightarrow \mathbf{R}$ *be differentiable functions. Suppose there exists* $x_0 \in [a, b]$ *such that the sequence* $(f_n(x_0))$ *converges, and that the sequence* (f_n') *converges uniformly to a function* $g : [a, b] \longrightarrow \mathbf{R}$*. Then the sequence* (f_n) *converges uniformly to a differentiable function* $f : [a, b] \longrightarrow \mathbf{R}$*. Moreover* $f' = g$*, in other words*

$$f'(c) = \lim_{n\to\infty} f_n'(c) \qquad \text{for all } c \in [a, b].$$

Theorem 6.31. *For each $n \in \mathbf{N}$, let $f_n : [a,b] \longrightarrow \mathbf{R}$ be a Riemann integrable function. Suppose that the sequence (f_n) converges uniformly to a function $f : [a,b] \longrightarrow \mathbf{R}$. Then f is Riemann integrable and*

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b f_n(x)\,dx.$$

6.3 Power series

Definition 6.32. Let (a_n) be a sequence in $\mathbf{R}$ and let $c \in \mathbf{R}$. The corresponding *power series at c* is given by

$$\sum_{n=0}^{\infty} a_n (x-c)^n = a_0 + a_1(x-c) + a_2(x-c)^2 + \dots$$

The set S of all $x \in \mathbf{R}$ for which the power series converges is called the *domain of convergence* of the power series.

Theorem 6.33. *If the power series $\sum_{n=0}^{\infty} a_n(x-c)^n$ converges at some point $x_0 \in \mathbf{R}$, then it converges absolutely at any $x \in \mathbf{R}$ satisfying $|x-c| < |x_0 - c|$.*

Corollary 6.34. *The domain of convergence of a power series at c is of one of the following forms:*

$$(c-R, c+R) \textit{ or } (c-R, c+R] \textit{ or } [c-R, c+R) \textit{ for some } R \in \mathbf{R}_{>0}, \quad [c-R, c+R] \textit{ for some } R \in \mathbf{R}_{\geqslant 0}, \quad \mathbf{R}.$$

The real number R is called the *radius of convergence* of the power series. In the case $\mathbf{R}$ we declare (by abuse of notation) $R = \infty$.

Because of the Corollary, the domain of convergence is also called *interval of convergence*.

We will write

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$$

to denote the function $f: E \longrightarrow \mathbf{R}$ that is the pointwise limit of the series, where E is the interval of convergence of the series.

Example 6.35.

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Lemma 6.36. *If the power series* $\sum_{n=0}^{\infty} a_n(x-c)^n$ *converges absolutely at* $x_0 \in \mathbf{R}$*, then it converges uniformly on the closed interval* $[c-r_0, c+r_0]$*, where* $r_0 = |x_0 - c|$.

Theorem 6.37. *Consider a power series with radius of convergence* $R > 0$*:*

$$f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n \qquad \text{for all } x \in (c-R, c+R).$$

(a) The function f *is differentiable on* $(c-R, c+R)$ *with derivative*

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x-c)^{n-1} \qquad \text{for all } x \in (c-R, c+R).$$

(b) The function f *is integrable on* (c, x) *for every* $x \in (c-R, c+R)$ *and*

$$\int_c^x f(t)\,dt = \sum_{n=0}^{\infty} \frac{a_n(x-c)^{n+1}}{n+1}.$$

Example 6.38. Going back to $f : \mathbf{R} \longrightarrow \mathbf{R}$ defined by

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Theorem 6.39 (Algebra of Power Series). *Let* $c \in \mathbf{R}$ *and*

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n, \qquad g(x) = \sum_{n=0}^{\infty} b_n (x-c)^n$$

be power series at c *with respective intervals of convergence* E_f *and* E_g.

Then

$$f(x) + g(x) = \sum_{n=0}^{\infty} (a_n + b_n)(x-c)^n \qquad \textit{for all } x \in E_f \cap E_g.$$

If $\lambda \in \mathbf{R}$, *then*

$$\lambda f(x) = \sum_{n=0}^{\infty} \lambda a_n (x-c)^n \qquad \textit{for all } x \in E_f.$$

Cauchy product (discrete convolution)

What about $f(x)g(x)$?

Before answering this question, let's go back to series of numbers and suppose $\sum_{n=0}^{\infty} a_n = \alpha$ and $\sum_{n=0}^{\infty} b_n = \beta$.

Is there something that has limit $\alpha\beta$?

The *Cauchy product* of infinite series $\sum_{n=0}^{\infty} a_n$ and $\sum_{n=0}^{\infty} b_n$ is the infinite series $\sum_{n=0}^{\infty} d_n$, where $d_n = \sum_{k=0}^{n} a_k b_{n-k}$.

A slightly more rigorous version of our above calculation establishes:

Theorem 6.40. *If* $\sum_{n=0}^{\infty} a_n = \alpha$ *and* $\sum_{n=0}^{\infty} b_n = \beta$ *and at least one of the two series converges absolutely, then their Cauchy product converges to* $\alpha\beta$*.*

The power series version is:

Theorem 6.41 (Product of Power Series)**.** *Let* $c \in \mathbf{R}$ *and*

$$f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n, \qquad g(x) = \sum_{n=0}^{\infty} b_n (x-c)^n$$

be power series at c *with respective intervals of convergence* E_f *and* E_g*.*

Then

$$f(x)g(x) = \sum_{n=0}^{\infty} d_n (x-c)^n \qquad \text{for all } x \in E_f \cap E_g,$$

where

$$d_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

Example 6.42. Going back once more to $f: \mathbf{R} \longrightarrow \mathbf{R}$ defined by

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Taylor series

Definition 6.43. Let $c \in (a, b)$ and let $f : (a, b) \longrightarrow \mathbf{R}$ be a function that is (at least) n times differentiable at c.

The n-th *Taylor polynomial* of f at c is

$$T_{n,c}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!}(x-c)^k = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n.$$

If $c = 0$ we write $T_n(x) = T_{n,0}(x)$.

Theorem 6.44. *We have*

$$T_{n,c}^{(k)}(c) = f^{(k)}(c) \qquad \textit{for all } 0 \leqslant k \leqslant n.$$

Definition 6.45. Let $c \in (a,b)$ and let $f:\ (a,b) \longrightarrow \mathbf{R}$ be a function that is (at least) n times differentiable at c.

The *Taylor series* of f at c is

$$\lim_{n\to\infty} T_{n,c}(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k.$$

Example 6.46. Suppose that you have defined the exponential function $f(x) = e^x$ in some way that gives you that $f(0) = 1$ and $f'(x) = e^x$.

We think of the Taylor polynomial $T_{n,c}$ as an approximation to the function f near the point c. The error in approximating the value $f(x)$ by $T_{n,c}(x)$ is

$$|f(x) - T_{n,c}(x)|.$$

Theorem 6.47 (Taylor)**.** *Let $c \in (a,b)$ and let $f:\ (a,b) \longrightarrow \mathbf{R}$ be a function that is (at least) $n+1$ times differentiable at c.*

For every $x \in (a,b) \setminus \{c\}$ there exists ξ between c and x such that

$$f(x) - T_{n,c}(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\,(x-c)^{n+1}.$$

Corollary 6.48. *Under the same assumptions as Taylor's Theorem, suppose there exists a constant $M > 0$ such that $|f^{(n+1)}(t)| \leqslant M$ for all t between c and x. Then*

$$|f(x) - T_{n,c}(x)| = \frac{M}{(n+1)!}\,|x-c|^{n+1}.$$

Example 6.49. Consider a function $f : \mathbf{R} \longrightarrow \mathbf{R}$ with the properties $f(0) = 1$ and $f'(x) = f(x)$ for all $x \in \mathbf{R}$. Prove that

$$f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} \qquad \text{for all } x \in \mathbf{R}.$$

Example 6.50. Explore the convergence of the Taylor series at 0 of the function $f: \mathbf{R} \longrightarrow \mathbf{R}$ defined by

$$f(x) = \begin{cases} e^{-1/x} & \text{if } x > 0, \\ 0 & \text{if } x \leqslant 0. \end{cases}$$

6.4 Fourier series

Definition 6.51. Let $f : \mathbf{R} \longrightarrow \mathbf{R}$ be integrable on $[-\pi, \pi]$ and (2π)-periodic, that is $f(x+2\pi) = f(x)$ for all $x \in \mathbf{R}$.

The *Fourier series* of f is

$$a_0 + \sum_{k=1}^{\infty} \big(a_k \cos(kx) + b_k \sin(kx)\big),$$

where

$$\begin{aligned} a_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx, \\ a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(kx)\, dx \quad \text{for } k \geqslant 1, \\ b_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(kx)\, dx \quad \text{for } k \geqslant 1. \end{aligned}$$

Example 6.52. Consider the (2π)-periodic function $f : \mathbf{R} \longrightarrow \mathbf{R}$ given by $f(x) = x$ for all $x \in (-\pi, \pi]$.

What is the relation between the values of the Fourier series and the values of f itself? In other words, when does the Fourier series of f converge to f?

Theorem 6.53. *Let $f : \mathbf{R} \longrightarrow \mathbf{R}$ be bounded and (2π)-periodic. Suppose that f is piecewise continuous and piecewise monotone, that is there is a partition*

$$-\pi = x_0 < x_1 < x_2 < \cdots < x_{n-1} < x_n = \pi$$

so that f is continuous and monotone on (x_{i-1}, x_i) for all $1 \leqslant i \leqslant n$.

Then the Fourier series converges pointwise to $f(x)$ for every $x \in \mathbf{R}$ at which f is continuous:

$$f(x) = a_0 + \sum_{k=1}^{\infty} \big(a_k \cos(kx) + b_k \sin(kx)\big).$$

If x is a point of discontinuity of f, the Fourier series converges to the average of the left and right limits at x:

$$f(x) = \frac{1}{2}\left(\lim_{t \to x^-} f(t) + \lim_{t \to x^+} f(t)\right).$$

Example 6.54. What is the limit of the Fourier series of the function from Example 6.52?

Example 6.55. Consider the (2π)-periodic function $f : \mathbf{R} \longrightarrow \mathbf{R}$ given by $f(x) = x^2$ for all $x \in (-\pi, \pi]$.