2 Set theory

Chapter contents

2.1 Sets

Definition 2.1. A *set* is a collection of unique objects, called *elements*. If A is a set and x is an element of A, we write $x \in A$. If x is not an element of A, we write $x \notin A$.

Example 2.2. $A = \big\{\mathbf{Z}, 0, \mathbf{Q}, \mathbf{R}, \{\pi, e, 1\}\big\}$ is a set.

We have: $\mathbf{Z} \in A$, $\mathbf{N} \notin A$, $0 \in A$, $1 \notin A$.

The elements of a set are unordered:

$$\big\{\mathbf{Z}, 0, \mathbf{Q}, \mathbf{R}, \{\pi, e, 1\}\big\} = \big\{0, \mathbf{Q}, \mathbf{Z}, \{\pi, e, 1\}, \mathbf{R}\big\}.$$

Most interesting sets are too big or too complicated to be described by a complete explicit enumeration of their elements.

Example 2.3. The set of multiples of 17:

$$S = \{x \in \mathbf{Z} : \ (\exists k \in \mathbf{Z}) x = 17k\} = \{17k : \ k \in \mathbf{Z}\} = \{\dots, -34, -17, 0, 17, 34, 51, \dots\}.$$

The set of Fermat primes:

$$S = \{x \in \mathbf{Z} : \ (\exists k \in \mathbf{Z}) x = 2^{2^k} + 1 \text{ and } x \text{ is prime}\} = \{3, 5, 17, 257, 65537, ???\}.$$

This type of description is called *set comprehension* and is close to the informal natural language description. It is very powerful, but with great power comes great responsibility.

Example 2.4. Let S be the set of all things that are not elephants.

Some elements of S: cat, dog, Alex, $\mathbf{Z}$, 1, π, S.

If you don't see a problem with the above, consider

Example 2.5 (Russell's Paradox)**.** Let P be the set of all sets that are not elements of themselves. Is $P \in P$?

- If $P \in P$, then P is an element of itself, to $P \notin P$ by the definition of P, contradiction.
- If $P \notin P$, then P is not an element of itself, so $P \in P$ by the definition of P, contradiction.

Example 2.6 (Barber Paradox)**.** There is a barber who shaves all those, and only those, who do not shave themselves. Does the barber shave himself?

Issues such as these paradoxes forced mathematicians to develop formal axiomatic descriptions of set theory.

ZFC axioms of set theory

The most commonly used formalisms are ZFC (Zermelo–Fraenkel with Choice) or one of its variants such as NBG (von Neumann–Bernays–Gödel).

Definition 2.7 (ZFC axioms)**.**

- **Extensionality:** two sets are equal if and only if they have the same elements.
- **Pairing:** for any a and b there exists a set $\{a, b\}$ that contains exactly a and b.
- **Separation:** if U is a set and $p(x)$ is a condition on U expressed in first-order logic, there exists a set $X = \{x \in U : \ p(x)\}$.
- **Union:** if A is a set of sets X_α, there exists a set $\bigcup_{\alpha \in A} X_\alpha$ that contains all elements that belong to at least one X_α.
- **Power set:** if X is a set, there exists a set $P(X)$ that contains all the subsets of X.
- **Inductive set:** there exists a set N such that every $x \in N$ is a set, $\varnothing \in N$, and if $x \in N$ then $x \cup \{x\} \in N$.
- **Replacement:** if f is a function, the range of f is a set.
- **Regularity:** if A is a non-empty set, there exists $y \in A$ such that A and y have no common elements.
- **Choice:** if A is a set of nonempty sets X_α, there exists a set that contains exactly one element from each set X_α.

Equality

The Axiom of Extensionality defines the notion of equality between two sets:

Definition 2.8. Two sets A and B are *equal* (we write: $A = B$) when

$$(\forall x) x \in A \Leftrightarrow x \in B.$$

This suggests the possibility of a weaker notion:

Definition 2.9. A set A is a *subset* of a set B (we write: $A \subseteq B$) when

$$(\forall x) x \in A \Rightarrow x \in B.$$

If $A \subseteq B$ and $A \neq B$ we write $A \subsetneq B$ and say that A is a *proper subset* of B.

Example 2.10. $X = \{1, 2, 3, 4, 5\}$.

$A = \{1, 2, 5\} \subseteq X$, $B = \{3, 1, 4\} \subseteq X$, $C = \{3, 7\} \not\subseteq X$ since $7 \in C$ but $7 \notin X$.

Separation

The Axiom of Separation formalises the correct use of set comprehension: if U is a set and $p(x)$ is a condition on U, there exists a set

$$X = \{x \in U : \; p(x) \text{ is } \mathsf{True}\}$$

consisting precisely of the elements x of U for which $p(x)$ is True.

Example 2.11. Take $U = \{1, 2, 3, 4, 5\}$ and $p(x) : \; x < 4$.

We have

$$X = \{x \in U : \; p(x)\} = \{x \in U : \; x < 4\} = \{1, 2, 3\}$$
$$Y = \{y \in U : \; \neg p(y)\} = \{y \in U : \; y \geqslant 4\} = \{4, 5\}.$$

Note that the Axiom of Separation implies the existence of the *empty set* $\varnothing = \{\}$, with no elements.

Take U to be any set (for instance the set N whose existence is stated in the Axiom of Inductive Set), then

$$\varnothing = \{x \in U : \; x \neq x\}.$$

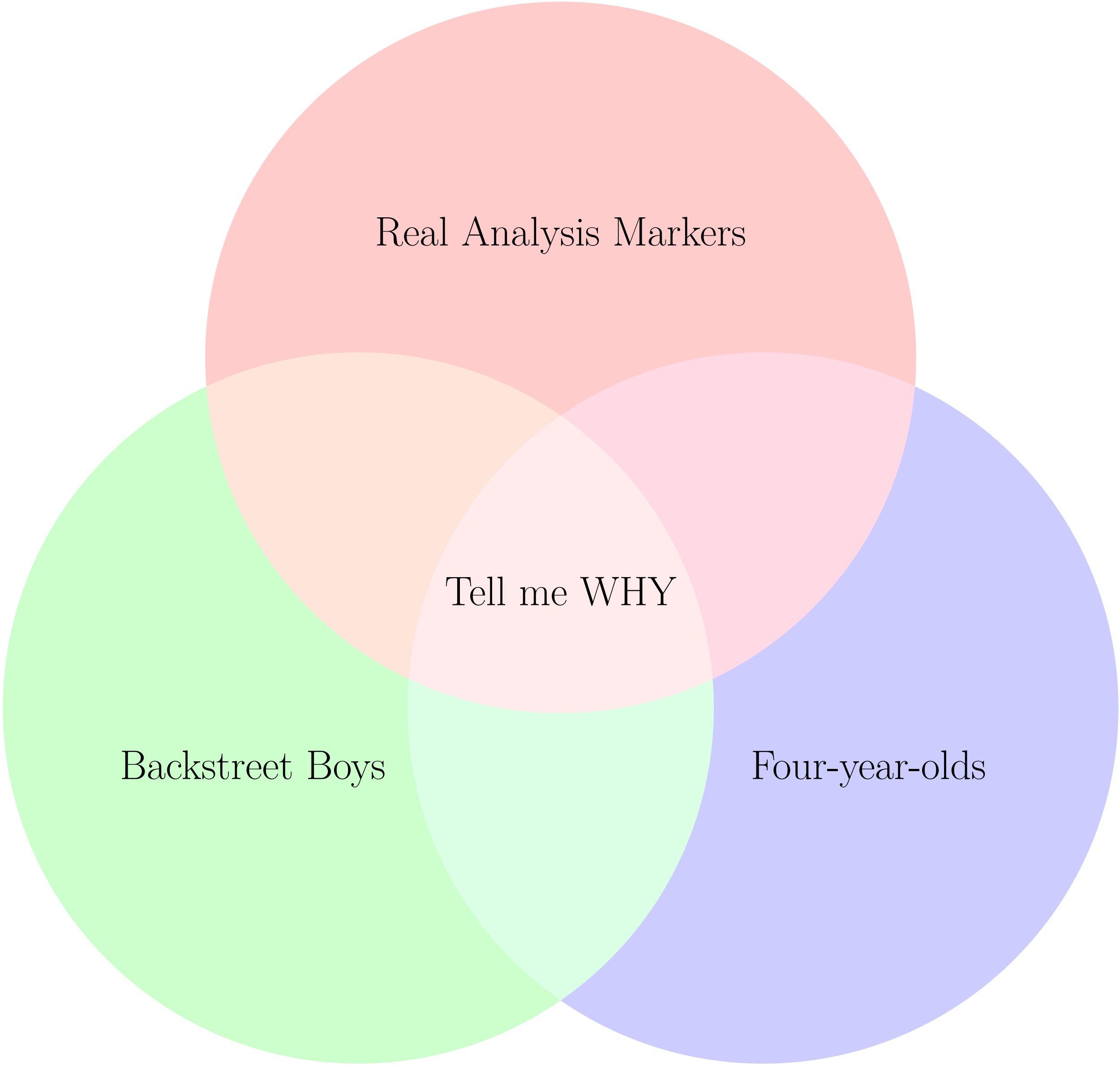
Operations with sets

Here we assume that all the sets we consider are subsets of a fixed set U.

Definition 2.12. Let A and B be sets.

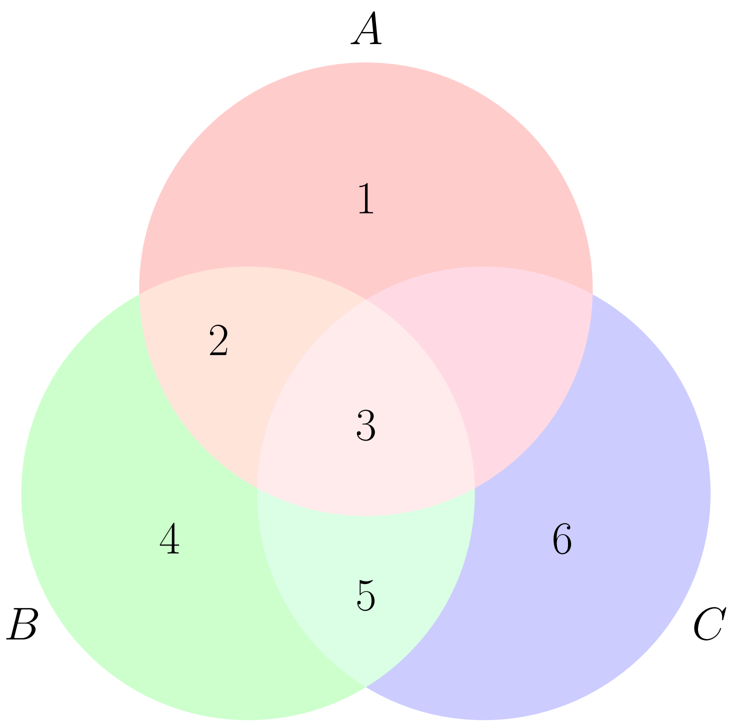
- the *union* of A and B is
$$A \cup B = \{x \in U : \ x \in A \text{ or } x \in B\}.$$
- the *intersection* of A and B is
$$A \cap B = \{x \in U : \ x \in A \text{ and } x \in B\}.$$
- the *difference* (sometimes called the *complement* of B in A) is
$$A \setminus B = \{x \in U : \ x \in A \text{ and } x \notin B\}.$$

Example 2.13 (Everybody ♥ Venn diagrams).



Example 2.14 (Everybody ♥ Venn diagrams)**.**

Let $A = \{1, 2, 3\}$, $\quad B = \{2, 3, 4, 5\}$, $\quad C = \{3, 5, 6\}$.



$$A \cup B = \{1, 2, 3, 4, 5\}$$
$$A \cap B = \{2, 3\}$$
$$A \setminus B = \{1\}$$
$$(A \cap C) \setminus B = \{\} = \varnothing.$$

Cartesian product

Let A and B be sets.

Definition 2.15. An *ordered pair* (a, b) is a set of the form

$$(a,b) = \{\{a\},\{a,b\}\}, \qquad \text{where } a \in A \text{ and } b \in B.$$

Note that the set (a, b) has two elements: $\{a\}$ and $\{a, b\}$; note also that $\{a\} \subseteq \{a, b\}$.

Example 2.16. $(1, 2) = \{\{1\}, \{1, 2\}\}$.

$(1, 2) \neq (2, 1)$ since $\{\{1\}, \{1, 2\}\} \neq \{\{2\}, \{2, 1\}\}$.

$(1, 1) = \{\{1\}, \{1, 1\}\} = \{\{1\}, \{1\}\} = \{\{1\}\}$.

The essential property of ordered pairs is (see Tutorial Question 4.5)

$$\text{given } a_1, a_2 \in A \text{ and } b_1, b_2 \in B, \text{ we have } (a_1, b_1) = (a_2, b_2) \text{ if and only if } a_1 = a_2 \text{ and } b_1 = b_2.$$

Let A and B be sets.

Definition 2.17. The *Cartesian product* of A and B is

$$A \times B = \big\{(a, b) : \; a \in A \text{ and } b \in B\big\}.$$

Example 2.18. Let $A = \{1, 5, \text{dog}\}$ and $B = \{5, \text{cat}\}$.

Some proofs with sets

Theorem 2.19. *Let A and B be sets. We have $A = B$ if and only if ($A \subseteq B$ and $B \subseteq A$).*

Proof. In one direction, suppose $A = B$.

Let $x \in A$, then (since $A = B$) we have $x \in B$. Therefore $A \subseteq B$.

Similarly, let $x \in B$, then (since $B = A$) we have $x \in A$. Therefore $B \subseteq A$.

We conclude that ($A \subseteq B$ and $B \subseteq A$).

For the other direction, suppose $A \subseteq B$ and $B \subseteq A$.

Let $x \in A$. Since $A \subseteq B$, we have $x \in B$. Therefore $x \in A \Rightarrow x \in B$.

Similarly, let $x \in B$. Since $B \subseteq A$, we have $x \in A$. Therefore $x \in B \Rightarrow x \in A$.

We conclude that $x \in A \Leftrightarrow x \in B$, hence $A = B$. □

Theorem 2.20. *Let A, B, and C be sets. Then $(B \setminus A) \cap C = (B \cap C) \setminus A$.*

Proof. We use Theorem 2.19.

First we show that $(B \setminus A) \cap C \subseteq (B \cap C) \setminus A$.

Let $x \in (B \setminus A) \cap C$. Then $x \in B \setminus A$ and $x \in C$. But if $x \in B \setminus A$ then $x \in B$ and $x \notin A$. Since $x \in B$ and $x \in C$, we have $x \in B \cap C$. Since $x \notin A$, we have $x \in (B \cap C) \setminus A$, as claimed.

Now we show that $(B \cap C) \setminus A \subseteq (B \setminus A) \cap C$.

Let $x \in (B \cap C) \setminus A$. Then $x \in B \cap C$ and $x \notin A$. Therefore $x \in B$ and $x \in C$. Since $x \in B$ and $x \notin A$, we have $x \in B \setminus A$. Since $x \in C$, we have $x \in (B \setminus A) \cap C$, as claimed. □

Numbers

We spend lots of time working with sets of numbers. Recall the most common ones:

The *natural numbers*

$$\mathbf{N} = \{0, 1, 2, 3, 4, \dots\}.$$

The existence of such a set is guaranteed by the Axiom of Inductive Set: “there exists a set N such that every $x \in N$ is a set, $\varnothing \in N$, and if $x \in N$ then $x \cup \{x\} \in N$.”

So we have

$$N = \{\varnothing, \{\varnothing\}, \{\varnothing, \{\varnothing\}\}, \dots\}$$

Informally, we can think of this as: $0 \in \mathbf{N}$, and if $n \in \mathbf{N}$ then $n+1 \in \mathbf{N}$. This should remind you of the Principle of Mathematical Induction!

Theorem 2.21 (Well-Ordering Property of $\mathbf{N}$). *Every non-empty subset $S \subseteq \mathbf{N}$ has a minimum: there exists $m \in S$ such that $m \leqslant x$ for all $x \in S$.*

You get to prove this in Tutorial Question 4.10.

Starting from the natural numbers $\mathbf{N}$, we successively broaden our numerical horizons:

- the *integers*
$$\mathbf{Z} = \{ \pm n : \ n \in \mathbf{N} \} = \{ \dots, -3, -2, -1, 0, 1, 2, 3, \dots \};$$

- the *rational numbers*
$$\mathbf{Q} = \left\{ \frac{a}{b} : \ a \in \mathbf{Z}, b \in \mathbf{Z}_{>0} \right\}, \quad \text{with } \frac{a}{b} = \frac{c}{d} \Leftrightarrow ad = bc;$$

- the *real numbers* $\mathbf{R}$, to be discussed in more depth soon;

- the *complex numbers*
$$\mathbf{C} = \{ a + ib : \ a \in \mathbf{R}, b \in \mathbf{R} \}, \quad \text{where } i^2 = -1.$$

We have $\mathbf{N} \subsetneq \mathbf{Z} \subsetneq \mathbf{Q} \subsetneq \mathbf{R} \subsetneq \mathbf{C}$.

We also have the standard addition and multiplication operations on these sets, with the usual arithmetic properties.

2.2 Relations, functions, orders

Definition 2.22. Let A and B be sets. A *relation* from A to B is a subset $R \subseteq A \times B$.

For $a \in A$, $b \in B$, we write aRb if $(a, b) \in R$.

Example 2.23. The University of Melbourne has a database of the students enrolled in various subjects.

Let P be the set of all students and S the set of all subjects. Then the University's enrollment database can be described mathematically as

$$E = \{(p, s) \in P \times S : \text{ student } p \text{ is enrolled in subject } s\} \subseteq P \times S.$$

You can "visualise" a relation R from A to B as a table with rows indexed by the elements of A, columns indexed by the elements of B, and True appearing in row a column b if and only if aRb.

We will soon talk about two special types of relations that are important to analysis: functions and orders.

It is worth noting that other mathematical concepts can be naturally viewed as relations.

Example 2.24 (Equality in a set). Let X be a set and consider the *diagonal subset* of $X \times X$, given by

$$\Delta(X) = \big\{(x,x) : \ x \in X\big\} = \big\{(x,y) \in X \times X : \ x = y\big\}.$$

This is the equality relation on X: $(x,y) \in \Delta(X) \Leftrightarrow x = y$.

Example 2.25 (Element of a set). Let X be a set and let $P(X)$ be its power set. Consider

$$R = \big\{(x,A) \in X \times P(X) : \ x \in A\big\}.$$

So "$x \in A$" is a relation from X to $P(X)$.

Example 2.26 (Subset). Let X be a set and let $P(X)$ be its power set. Consider

$$R = \big\{(A,B) \in P(X) \times P(X) : \ A \subseteq B\big\}.$$

So "$A \subseteq B$" is a relation on $P(X)$.

Functions

Definition 2.27. Let A and B be sets; let $f \subseteq A \times B$ be a relation from A to B.

We say that f is a *function* from A to B if: for every $x \in A$ there exists a unique $y \in B$ such that $(x, y) \in f$.

We write $f: A \longrightarrow B$; A is called the *domain* of f; B is called the *codomain* of f.

If $(x, y) \in f$ then we write $y = f(x)$; y is called the *image* of x. The set

$$f^{-1}(y) = \left\{x \in A : f(x) = y\right\} \subseteq A$$

is called the *preimage* of y.

More generally, if $S \subseteq A$ then the set

$$f(S) = \left\{y \in B : (\exists x \in S) f(x) = y\right\} \subseteq B$$

is called the *image* of S. In the special case $S = A$, the set $f(A)$ is called the *image* (or *range*) of f.

If $T \subseteq B$ then the set

$$f^{-1}(T) = \left\{x \in A : f(x) \in T\right\} \subseteq A$$

is called the *preimage* of T.

Example 2.28. Let $f: \mathbf{R} \longrightarrow \mathbf{R}$ be the function defined by $f(z) = z^2$ for all $z \in \mathbf{R}$.

Draw the graph of this function.

We have $(-2, 4) \in f$, in other words $f(-2) = 4$. Therefore 4 is the image of -2.

The preimage of 4 is $\{-2, 2\}$. Consider the set $S = [-1, 4] \subseteq \mathbf{R}$. The image of S is $f(S) = [0, 16]$.

Consider the set $T = [-1, 4] \subseteq \mathbf{R}$. The preimage of T is $f^{-1}(T) = [-2, 2]$.

The preimage of $[-3, -1]$ is $f^{-1}([-3, -1]) = \varnothing$.

The image of f is $f(\mathbf{R}) = [0, \infty)$.

Definition 2.29. Let A and B be sets and let $f : A \longrightarrow B$ be a function. We say that f is ...

- ... *injective* if $(\forall x, y \in A)\ \big(f(x) = f(y) \Rightarrow x = y\big)$;
- ... *surjective* if $(\forall z \in B)\ (\exists x \in A)\ f(x) = z$;
- ... *bijective* if f is both injective and surjective.

Example 2.30. The function $f : \mathbf{R} \longrightarrow \mathbf{R}$ given by $f(x) = e^x$ is injective, but not surjective.

Example 2.31. The function $f : \mathbf{R} \longrightarrow \mathbf{R}_{\geqslant 0}$ given by $f(x) = x^2$ is surjective, but not injective.

Example 2.32 (Identity function)**.** For any set A, there exists a function id_A, called the *identity function on A*, defined by

$$\mathrm{id}_A(x) = x \qquad \text{for all } x \in A.$$

It is easily seen to be bijective.

Example 2.33. Most functions are neither injective nor surjective, for instance $f : \mathbf{R} \longrightarrow \mathbf{R}$ given by $f(x) = \sin x$.

The most fundamental operation on functions is *composition*: if $f : A \longrightarrow B$ and $g : B \longrightarrow C$, then the composition $g \circ f$ is the function $A \longrightarrow C$ defined by

$$(g \circ f)(x) = g(f(x)) \qquad \text{for all } x \in A.$$

If $A = B$, we have $g \circ \mathrm{id}_A = g$ for all functions $g : A \longrightarrow C$.

If $B = C$, we have $\mathrm{id}_B \circ f = f$ for all functions $f : A \longrightarrow B$.

Definition 2.34. Let A and B be sets. We say that a function $f : A \longrightarrow B$ is *invertible* if there exists a function $g : B \longrightarrow A$ such that

$$g \circ f = \mathrm{id}_A \qquad \text{and} \qquad f \circ g = \mathrm{id}_B\,.$$

Theorem 2.35. *Let $f : A \longrightarrow B$ be a function from a set A to a set B. Then f is bijective if and only if it is invertible.*

See Exercise 2.8.

Orders

Definition 2.36. Let S be a set. A *(total) order* on S is a relation $<$ satisfying the following two axioms:

(**O1**) for all $x, y \in S$, exactly one of the following statements is True: $x < y$, $y < x$, $x = y$;

(**O2**) for all $x, y, z \in S$, if $x < y$ and $y < z$, then $x < z$.

An *ordered set* S is a set with an order $<$.

We write $x > y$ to mean the statement $y < x$.

We write $x \leqslant y$ and $y \geqslant x$ to mean the statement $(x < y) \vee (x = y)$.

Example 2.37. $S = \mathbf{R}$ (or $\mathbf{Q}$ or $\mathbf{Z}$ or $\mathbf{N}$) with the usual inequality relation $<$ is an ordered set.

Example 2.38. A *singleton* (set with a single element) $S = \{a\}$ is ordered via

the empty order $<= \varnothing \subseteq S \times S$.

Example 2.39. Let $S = \{a, b, c\}$. Which of the following relations is an order on S?

- $a < b$, $a < c$, $b < c$;

  Yes: $a < b < c$.

- $b < a$, $c < a$, $b < c$;

  Yes: $b < c < a$.

- $a < b$, $b < c$, $c < a$.

  No: $a < b$ and $b < c$ implies $a < c$, contradicting $c < a$.

Theorem 2.40. *Let S be an ordered set. If $x, y \in S$ are such that $x \leqslant y$ and $y \leqslant x$, then $x = y$.*

Proof. We proceed by contradiction.

Suppose $x \neq y$. Then by axiom (O1) we must have $x < y$ or $y < x$.

- If $x < y$, then both statements $y < x$ and $y = x$ are False, hence $y \leqslant x$ is False, contradiction.
- Similarly, if $y < x$, then both statements $x < y$ and $x = y$ are False, hence $x \leqslant y$ is False, contradiction. □

2.3 The real numbers

The most important sets in real analysis are arguably the subsets of $\mathbf{R}$, the set of real numbers.

This set has some interesting properties coming in two intermingling flavours:

- algebraically, it is a field under the usual operations of addition and multiplication;
- it has an order relation $<$.

Moreover, $\mathbf{R}$ contains the field of rational numbers $\mathbf{Q}$ in a way that is compatible with both the algebra and the order relation.

We will not dwell on the algebraic aspects: you are familiar with them, and they are better discussed in algebra subjects.

TODO: mention uniqueness?

The properties of the order relation are crucial in analysis, so we will focus on these.

Some are given as axioms:

(**RO1**) for all $x, y \in \mathbf{R}$, exactly one of the following is True:

$$x < y, \qquad y < x, \qquad x = y;$$

(**RO2**) for all $x, y, z \in \mathbf{R}$, if $x < y$ and $y < z$ then $x < z$;

(**RO3**) for all $x, y, c \in \mathbf{R}$, if $x < y$ then $x + c < y + c$;

(**RO4**) for all $x, y \in \mathbf{R}$, if $0 < x$ and $0 < y$ then $0 < xy$;

(**RO5**) Completeness Axiom: every non-empty subset of $\mathbf{R}$ that is bounded above in $\mathbf{R}$ has a supremum in $\mathbf{R}$.

Other properties of the order relation can be proved from the axioms.

Theorem 2.41. *Let $x, y, z \in \mathbf{R}$ with $0 < z$. If $x < y$ then $xz < yz$.*

Proof. We know that $x < y$. Add $(-x)$ to both sides using axiom (RO3) to get $0 < y - x$. Since we are told that $0 < z$, we can multiply both sides by z using axiom (RO4) to get $0 < yz - xz$. Now add xz to both sides using axiom (RO3) to get $xz < yz$. □

Theorem 2.42. *Let $x, y \in \mathbf{R}$. If $x < y$ then $-y < -x$.*

See Exercise 2.27.

Theorem 2.43. *Let $x, y \in \mathbf{R}$. If $x < y$, then*

$$x < \frac{x+y}{2} < y.$$

See Exercise 2.27.

So far we have dealt with the order relation given by the strict inequality $x < y$. You should think about which of/whether the properties described above also hold for the non-strict inequality $x \leqslant y$.

We have not yet discussed the Completeness Axiom of $\mathbf{R}$. This requires a bit of terminology.

2.4 Bounds

Let S be an ordered set (e.g. $S = \mathbf{R}$) and let A be a subset of S.

Definition 2.44. We say that A is *bounded below* in S if there exists $\beta \in S$ such that

$$(\forall x \in A)\beta \leqslant x.$$

We say that β is a *lower bound* for A in S.

We say that A is *bounded above* in S if there exists $\beta \in S$ such that

$$(\forall x \in A)x \leqslant \beta.$$

We say that β is an *upper bound* for A in S.

We say that A is *bounded* in S if it is bounded both above and below in S.

Example 2.45. Let $S = \mathbf{R}$ and consider the subset

$$A = \left\{ \frac{1}{r} : \ r \in \mathbf{R} \text{ and } r > 0 \right\}.$$

$\beta = 0$ is a lower bound for A in $\mathbf{R}$.

$\beta = -100$ is also a lower bound for A in $\mathbf{R}$.

In fact, every element of $\mathbf{R}_{\leqslant 0}$ is a lower bound for A in $\mathbf{R}$.

A is not bounded above in $\mathbf{R}$. In other words, the set of upper bounds for A in $\mathbf{R}$ is $\varnothing$.

We prove this by contradiction. Suppose $\alpha \in \mathbf{R}$ is an upper bound for A. Since $1 = \frac{1}{1} \in A$, we have $\alpha \geqslant 1$.

Let $r = \frac{1}{2\alpha}$, then $r \in \mathbf{R}$ and $r > 0$. We have

$$2\alpha = \frac{1}{r} \leqslant \alpha \qquad \text{so} \qquad 2 \leqslant 1,$$

contradiction.

In the example, both -100 and 0 are lower bounds for A in $\mathbf{R}$, but there is a clear sense in which 0 is more precise than -100.

Let S be an ordered set and let A be a subset of S.

Definition 2.46. Let L_A be the set of all lower bounds for A in S. We say that $\alpha \in L_A$ is an *infimum* (or *greatest lower bound*) for A in S if

$$(\forall \beta \in L_A)\beta \leqslant \alpha.$$

Let U_A be the set of all upper bounds for A in S. We say that $\alpha \in U_A$ is a *supremum* (or *least upper bound*) for A in S if

$$(\forall \beta \in U_A)\alpha \leqslant \beta.$$

Theorem 2.47. *Let A be a subset of an ordered set S.*

- *If a supremum for A in S exists, it is unique.*
- *If an infimum for A in S exists, it is unique.*

Because of this uniqueness property, it is unambiguous to write $\sup A$ for **the** supremum of A, and $\inf A$ for **the** infimum of A.

Proof. We prove the statement for the supremum; the infimum statement is proved in an analogous way.

Let $\alpha_1, \alpha_2 \in S$ be suprema for A in S.

Therefore $\alpha_1 \in U_A$, and since α_2 is a supremum we have $\alpha_2 \leqslant \alpha_1$.

Similarly, $\alpha_2 \in U_A$, and since α_1 is a supremum we have $\alpha_1 \leqslant \alpha_2$.

From Theorem 2.40 we conclude that $\alpha_1 = \alpha_2$. □

Example 2.48. Revisiting

$$A = \left\{ \frac{1}{r} : \ r \in \mathbf{R} \text{ and } r > 0 \right\},$$

we have $\inf A = 0$. Here's why:

We have already noted that 0 is a lower bound for A in $\mathbf{R}$. Suppose there exists a lower bound β for A such that $\beta > 0$. Let $r = \frac{2}{\beta}$, then $r > 0$, so $\frac{1}{r} \in A$, but $\frac{1}{r} = \frac{\beta}{2} < \beta$, contradicting the fact that β is a lower bound for A.

Example 2.49. Consider
$$A = \{x \in \mathbf{R} : \ 1 \leqslant x < 2\}.$$

- We claim that $\inf A = 1$.

  It is clear from the definition of A that $1 \leqslant x$ for all $x \in A$, so 1 is a lower bound for A in $\mathbf{R}$, hence $\inf A \geqslant 1$.

  On the other hand $\inf A$ is a lower bound for A, and $1 \in A$, so $\inf A \leqslant 1$.

  We conclude that $\inf A = 1$.

- We claim that $\sup A = 2$.

  It is clear from the definition of A that $x < 2$ for all $x \in A$, so 2 is an upper bound for A in $\mathbf{R}$.

  Suppose that $2 \neq \sup A$. Then $2 > \sup A$. Let $\varepsilon = 2 - \sup A > 0$. Therefore $\sup A = 2 - \varepsilon < 2$. Let $x = 2 - \frac{\varepsilon}{2}$, then $\sup A = 2 - \varepsilon < x < 2$ and $x \in A$, contradiction.

We can turn the argument in the previous example into a more general result:

Theorem 2.50. *Let $A \subseteq \mathbf{R}$ be non-empty.*

(a) Suppose A is bounded above and let $\gamma \in \mathbf{R}$ be an upper bound for A in $\mathbf{R}$. Then $\gamma = \sup A$ if and only if: for every $\varepsilon > 0$ there exists $x \in A$ such that $x > \gamma - \varepsilon$.

(b) Suppose A is bounded below and let $\gamma \in \mathbf{R}$ be a lower bound for A in $\mathbf{R}$. Then $\gamma = \inf A$ if and only if: for every $\varepsilon > 0$ there exists $x \in A$ such that $x < \gamma + \varepsilon$.

Proof. We prove part (a); part (b) is proved by a similar argument.

Suppose $\gamma = \sup A$. Let $\varepsilon > 0$. Then $\gamma - \varepsilon < \gamma$, so $\gamma - \varepsilon$ is not an upper bound for A; therefore there exists $x \in A$ such that $x > \gamma - \varepsilon$.

For the other direction, suppose that for every $\varepsilon > 0$ there exists $x \in A$ such that $x > \gamma - \varepsilon$. We proceed by contradiction: suppose $\gamma \neq \sup A$. Then $\gamma > \sup A$. Letting $\varepsilon = \gamma - \sup A$, we have $\varepsilon > 0$. Then there exists $x \in A$ such that

$$x > \gamma - \varepsilon = \sup A,$$

contradiction. □

We now have everything we need to make sense of the

Completeness Axiom. Every non-empty subset $A \subseteq \mathbf{R}$ that is bounded above in $\mathbf{R}$ has a supremum in $\mathbf{R}$.

The analogous statement for infimum also holds, and follows from the one for the supremum, see Exercise 2.18.

There is an equivalent but more generalisable characterisation of completeness in terms of convergence of sequences, which we will discuss in the next chapter. It will, in particular, allow us to see that the equation $x^2 = 2$ has a positive real root $\sqrt{2}$.

The Archimedean Principle(s)

Theorem 2.51 (Archimedean Principle I)**.** *The subset* $\mathbf{N} \subseteq \mathbf{R}$ *is not bounded above in* $\mathbf{R}$.

Proof. We proceed by contradiction.

Suppose that $\mathbf{N}$ is bounded above in $\mathbf{R}$. Since $\mathbf{N}$ is non-empty, the Completeness Axiom says that $\alpha = \sup \mathbf{N}$ exists in $\mathbf{R}$. Since $-1 < 0$ we have $\alpha - 1 < \alpha$, so by the minimality of α we get that $\alpha - 1$ is not an upper bound of $\mathbf{N}$. So there exists $n \in \mathbf{N}$ such that $\alpha - 1 < n$.

But then $\alpha < n + 1$ and $n + 1 \in \mathbf{N}$, contradicting the fact that α is an upper bound of $\mathbf{N}$ in $\mathbf{R}$. □

But wait, there's more!

Theorem 2.52 (Archimedean Principle II)**.** *For every* $x \in \mathbf{R}$ *with* $x > 0$ *there exists* $n \in \mathbf{N}$ *such that* $n - 1 \leqslant x < n$.

Theorem 2.53 (Archimedean Principle III)**.** *If* $y, z \in \mathbf{R}$ *with* $y, z > 0$ *there exists* $n \in \mathbf{N}$ *such that* $y < nz$.

Theorem 2.54. *The three variants of the Archimedean Principle are pairwise equivalent.*

Proof. **(I $\Rightarrow$ II):** Let $x \in \mathbf{R}$. Since $\mathbf{N}$ is not bounded above, x is not an upper bound for $\mathbf{N}$. Therefore the set

$$S = \{m \in \mathbf{N} : \; m > x\}$$

is a non-empty subset of $\mathbf{N}$, hence by the Well-Ordering Property of $\mathbf{N}$, S has a minimal element n. Then $n > x$, and $n - 1 \notin S$, so $x \geqslant n - 1$.

(II $\Rightarrow$ III): Let $y, z \in \mathbf{R}$ with $y, z > 0$. Let $x = \frac{y}{z}$, then $x > 0$ so there exists $n \in \mathbf{N}$ such that $n - 1 \leqslant x < n$, hence $y < nz$.

(III $\Rightarrow$ I): Suppose $\mathbf{N}$ is bounded above in $\mathbf{R}$. Let y be an upper bound and let $z = 1$. Then there exists $n \in \mathbf{N}$ such that $y < n$, contradicting the fact that y is an upper bound of $\mathbf{N}$. $\square$

2.5 Distance on the real line

An important class of subsets of $\mathbf{R}$ consists of the *intervals*:

Definition 2.55. Given $a, b \in \mathbf{R}$ with $a \leqslant b$, we have

- the *closed interval* $[a, b] = \{x \in \mathbf{R} : \ a \leqslant x \leqslant b\}$;
- the *open interval* $(a, b) = \{x \in \mathbf{R} : \ a < x < b\}$;
- $(a, b] = \{x \in \mathbf{R} : \ a < x \leqslant b\}$;
- $[a, b) = \{x \in \mathbf{R} : \ a \leqslant x < b\}$;
- $(a, \infty) = \{x \in \mathbf{R} : \ a < x\}$;
- $[a, \infty) = \{x \in \mathbf{R} : \ a \leqslant x\}$;
- $(-\infty, b) = \{x \in \mathbf{R} : \ x < b\}$;
- $(-\infty, b] = \{x \in \mathbf{R} : \ x \leqslant b\}$.

Recall the *absolute value* function $\mathbf{R} \longrightarrow \mathbf{R}$ defined by

$$|x| = \begin{cases} -x & \text{if } x < 0, \\ x & \text{if } x \geqslant 0. \end{cases}$$

Definition 2.56. Given $x, y \in \mathbf{R}$, the *distance* between the points x and y on the real line is $|x - y|$.

Theorem 2.57. *Let $x, a \in \mathbf{R}$ with $a \geqslant 0$. We have*

$$|x| \leqslant a \qquad \text{if and only if} \qquad -a \leqslant x \leqslant a,$$

and similarly for the strict inequality $|x| < a$.

Proof. We prove the statement for the non-strict inequality $|x| \leqslant a$.

Suppose $|x| \leqslant a$.

- If $x < 0$, $|x| \leqslant a$ means $-x \leqslant a$, so $-a \leqslant x < 0 \leqslant a$.
- If $x \geqslant 0$, $|x| \leqslant a$ means $x \leqslant a$, so $-a \leqslant 0 \leqslant x \leqslant a$.

In all cases we get $-a \leqslant x \leqslant a$.

In the other direction, suppose $-a \leqslant x \leqslant a$.

- If $x < 0$, then $|x| = -x \leqslant a$ by multiplying by (-1) in $-a \leqslant x$.
- If $x \geqslant 0$, then $|x| = x \leqslant a$.

In all cases we get $|x| \leqslant a$. □

Theorem 2.58. *For every $x \in \mathbf{R}$ we have*

$$-x \leqslant |x| \qquad \text{and} \qquad x \leqslant |x|.$$

See Exercise 2.28.

Theorem 2.59 (The Triangle Inequality)**.** *For every $x, y \in \mathbf{R}$ we have*

$$|x+y| \leqslant |x| + |y|.$$

Proof. If $x + y \geqslant 0$, applying Theorem 2.58 twice we have

$$|x+y| = x + y \leqslant |x| + y \leqslant |x| + |y|.$$

Similarly, if $x + y < 0$, applying Theorem 2.58 twice we have

$$|x+y| = -(x+y) = -x - y \leqslant |x| - y \leqslant |x| + |y|.$$

□

Theorem 2.60. *For every $x, y \in \mathbf{R}$, if $x < y$ there exists $r \in \mathbf{Q}$ such that $x < r < y$.*

Proof. Writing $r = \frac{a}{b}$, it suffices to prove that

(a) there exists $b \in \mathbf{N}$ such that $b \geqslant 1$ and $|by - bx| \geqslant 2$;

(b) if $t, z \in \mathbf{R}$ with $t < z$ and $|z - t| \geqslant 2$, then there exists $a \in \mathbf{Z}$ such that $t < a < z$.

Indeed, if these two claims hold, then

$$bx = t < a < z = by \qquad \text{hence} \qquad x < \frac{a}{b} < z.$$

So it remains to prove the two claims.

(a) Since $x < y$ we have $y - x > 0$, so $\frac{2}{y-x} > 0$. Using the Archimedean Principle II, there exists $b \in \mathbf{N}$ such that $b \geqslant \frac{2}{y-x}$, so that $|by - bx| = by - bx \geqslant 2$, as needed.

(b) If $t > 0$, apply the Archimedean Principle II to get $a \in \mathbf{N}$ such that $a - 1 \leqslant t < a$. So $a - t \leqslant 1 < 2 \leqslant z - t$, hence $a < z$ and we conclude that $t < a < z$.

If $t \leqslant 0$, apply the Archimedean Principle II to $-t$ to get $c \in \mathbf{N}$ such that $c - 1 \leqslant -t < c$. Note that $c + t > 0$. Let $a = 2 - c$, then $a = 2 - c > 1 - c \geqslant t$, and $a - t = 2 - c - c < 2 \leqslant z - t$, so $a < z$. In conclusion $t < a < z$. □

Corollary 2.61. *For every* $z \in \mathbf{R}$ *and every* $\varepsilon > 0$, *there exists* $r \in \mathbf{Q}$ *such that* $|z - r| < \varepsilon$.

(A *corollary* is a theorem that follows easily from another theorem.)

Proof. Apply Theorem 2.60 with $x = z - \varepsilon$ and $y = z + \varepsilon$. □

A subset of **R** with the property given in the Corollary is said to be *dense* in **R**. So we have shown that **Q** is dense in **R**.