Topics: connectives, truth tables, tautology, contradiction, equivalence

Take a moment to write down the truth tables for the following connectives:

- negation $\neg p$
- conjunction $p \wedge q$
- disjunction $p \vee q$
- implication $p \Rightarrow q$
- biconditional $p \Leftrightarrow q$.

2.1 (Truth tables)**.** Using truth tables, for each of the compound statements below determine all possible truth values of p and q that make the statement True.

(a) $[p \wedge (p \Rightarrow q)] \Rightarrow q$

(b) $p \Rightarrow (q \wedge \neg q)$

(c) $(p \Rightarrow \neg q) \wedge (p \wedge q)$

(d) $[(p \vee q) \Rightarrow r] \wedge (r \Rightarrow \neg p)$.

When statement (b) is True, what can you conclude about p?

2.2 (Converse and contrapositive)**.**

(a) Suppose the following two statements are True:

- $p \Rightarrow q$
- p

i. Which row(s) of the truth table for implication corresponds to this case?

ii. What can we conclude about the truth value of q?

iii. Using sentences, give an example from your day-to-day life for when you can conclude something about a statement q given that both p and $p \Rightarrow q$ are True.

(b) Suppose the following two statements are True:

- $p \Rightarrow q$
- q

i. Which row(s) of the truth table for implication corresponds to this case?

ii. What can we conclude about the truth value of p?

iii. Using sentences, give an example from your day-to-day life for when you can't conclude something about a statement p given that both q and $p \Rightarrow q$ are True.

(c) Suppose the following two statements are True:

- $p \Rightarrow q$
- $\neg p$

i. Which row(s) of the truth table for implication corresponds to this case?

ii. What, if anything, can we conclude about the truth value of q?

iii. Using sentences, give an example from your day-to-day life for when you can't conclude something about a statement q given that both $\neg p$ and $p \Rightarrow q$ are True.

(d) Suppose the following two statements are True:

- $p \Rightarrow q$
- $\neg q$

i. Which row(s) of the truth table for implication corresponds to this case?

ii. What, if anything, can we conclude about the truth value of p?

iii. Using sentences, give an example from your day-to-day life for when you can conclude something about a statement p given that q is False and $p \Rightarrow q$ is True.

2.3 (Tautology, contradiction, and logical equivalence)**.**

(a) Give an example of a compound statement that is a tautology. Give an example of a compound statement that is a contradiction. The terms are introduced in Definition 1.17 in the lectures.

(b) Determine if the following statement is a tautology: $[(p \vee q) \wedge (\neg p)] \Rightarrow q$.

(c) With an English sentence, explain what is meant by the notation $r \equiv s$.

(d) Show that $p \Rightarrow q \equiv (\neg p) \vee q$.

2.4 (Kaching!)**.** Though the connectives we have introduced so far correspond to use in Australian English (not, or, and, etc...), we can use a truth table to define other connectives. For example, let \$ be the connective (that we will call *kaching!*) given by the following truth table:

p	q	$p \,\$\, q$
T	T	F
T	F	F
F	T	T
F	F	F

(a) Is $p \,\$\, p$ a tautology? a contradiction? both? neither?

(b) Is the kaching! connective commutative? In other words, is $p \,\$\, q \equiv q \,\$\, p$?

(c) Which of the statements below are logically equivalent to $[(\neg p) \,\$\, p] \,\$\, [q \,\$\, (\neg q)]$:

- $\neg p \Rightarrow q$
- $\neg(p \vee q)$
- $p \vee (\neg q)$
- $\neg p$

Topics: condition vs statement, universal and existential quantifiers, English to logic and vice versa

2.5 (Statements and conditions)**.** For each expression below, determine if it is a statement or a condition.

- For each condition give the values of the domain for which it is True/False.
- For each statement, determine if it is True or False. Justify your response by writing a sentence.

(a) $2x + 1 \geqslant 3$, for $x \in \mathbf{R}$

(b) $(\exists x \in \mathbf{R})\ 2x + 1 \geqslant 3$

(c) $(\forall x \in \mathbf{R})\ 2x + 1 \geqslant 3$

(d) $x^2 < 0$, for $x \in \mathbf{R}$

(e) $(\exists x \in \mathbf{R})\ x^2 < 0$

(f) $(\forall x \in \mathbf{R})\ x^2 < 0$

(g) $xy = y$, for $x, y \in \mathbf{R}$

(h) $(\forall x, y \in \mathbf{R})\ xy = y$

(i) $(\exists x, y \in \mathbf{R})\ xy = y$

(j) $(\forall x \in \mathbf{R})[(\exists y \in \mathbf{R})\ xy = y]$

(k) $(\exists x \in \mathbf{R})[(\forall y \in \mathbf{R})\ xy = y]$.

2.6 (Translating into the language of mathematics)**.** Translate the following sentences into formal logic. Which ones are True, which ones are False, and which ones are neither? If the sentence is ambiguous, explain why.

(a) "There is a rational number whose square is 7."

(b) "All even numbers are positive."

(c) "All subsets of the real numbers contain 12."

(d) "There are no real solutions of the equation $x^4 + 1 = 0$."

(e) "Every square root of 4 is positive."

(f) "The product of two integers is sometimes zero."

(g) "Some number is a multiple of both 6 and 10."

(h) "a is a multiple of n."

2.7 (Prime numbers)**.** A *prime number* is a natural number with exactly two positive divisors. For example: 7 is a prime number. (If d is a divisor of 7, then $d = 1$ or $d = 7$.)

(a) Express the condition "p is a prime number" using the language of formal logic. Use the notation $a \mid n$, which means "a is a divisor of n".

(b) Let P be the set of prime numbers. Express the condition "every integer greater than 1 that is not prime has at least one prime divisor" using the language of formal logic.

2.8. Negate each of these statements, then pull the negation as far to the right as possible. (If you get stuck, first write the statement as an English sentence.)

(a) $(\forall x \in S)\ \ q(x)$

(b) $(\forall x \in S)\ \ [(\exists y \in S)\ \ p(x, y)]$

(c) $(\exists x \in S)\ \ [(\forall y \in S)\ \ p(x, y)]$

(d) $(\exists x \in S)\ \ (q(x) \Rightarrow r(x))$.

Let $p(x, y)$ be the statement "$x + y = 0$", and let $S = \mathbf{R}$. Are the non-negated versions of (b) and (c) saying the same thing? Are they both True?

2.9 (Challenge: one connective to rule them all)**.** It turns out that one can express all the connectives in the set $\{\neg, \wedge, \Rightarrow, \vee, \Leftrightarrow\}$ in terms of the smaller set of connectives $\{\neg, \vee\}$.

(a) Show $p \wedge q \equiv \neg(\neg p \vee \neg q)$.

(b) Show $p \Rightarrow q \equiv (\neg p) \vee q$.

(c) Show $p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p)$.

(d) Find a statement that uses only $\neg$ and $\vee$ and is logically equivalent to $p \Leftrightarrow q$.

This motivates the following definition: Let C be a set of connectives. We say that C is *functionally complete* when every connective in the set $\{\neg, \wedge, \Rightarrow, \vee, \Leftrightarrow\}$ is logically equivalent to a compound statement that uses only connectives in the set C.

Our work in (a)–(d) shows that $\{\neg, \vee\}$ is functionally complete.

(e) Determine if there is a functionally complete set $\{\star\}$ containing a single element.

[**Hint**: Try to build the truth table for $\star$. You win if you can express each of $\neg$ and $\vee$ in terms of $\star$. A reasonable guess would be that $p \star p \equiv \neg p$, so start with this.]

p	q	$p \star q$
T	T	?
T	F	?
F	T	?
F	F	?