**Topic: proof techniques**

**3.1** (Counterexamples)**.** Using a counterexample, show that each of the following statements is False.

(a) Every natural number can be written as the sum of two perfect squares.

(b) Every quadrilateral with perpendicular diagonals has equal sides.

(c) If $f: \mathbf{R} \longrightarrow \mathbf{R}$ is continuous at $x = 0$, then $f$ is differentiable at $x = 0$.

**Note:** a correct solution requires a **specific** counterexample for each statement.

**3.2** (Existence proofs)**.** Express each of the following in the language of formal logic. Then provide a proof for each statement.

(a) Some integer $q$ satisfies $2q^2 - 9q = 5$.

(b) There is a natural number one more than a perfect square and one less than a perfect cube.

**Note:** a correct solution requires a **specific** example for each statement.

**3.3** (An integer is even if and only if its square is even)**.** In Theorem 1.36 we proved:

"Let $x$ be an integer. If $x$ is even, then $x^2$ is even."

(a) Write out the converse of the above statement.

(b) Write out the contrapositive of your statement from (a).

(c) Prove your statement from (b) with a direct proof.

(d) Using your work from parts (a)–(c) and the fact that

$$(p \Leftrightarrow q) \equiv [(p \Rightarrow q) \wedge (\neg p \Rightarrow \neg q)],$$

convince yourselves that the following statement is True

"An integer $x$ is even if and only if $x^2$ is even."

**3.4** (The square root of 2 is irrational)**.** We want to prove that $\sqrt{2} \notin \mathbf{Q}$. As this is stated, it supposes that we know what $\sqrt{2}$ is and where it belongs, namely the real numbers $\mathbf{R}$. Of course we do know this, but we have not yet proved it in this subject.

To avoid this issue, we can rewrite our statement as: "$(\forall x \in \mathbf{Q}) x^2 \neq 2$." This is what we prove now.

(a) Fill in the blanks in the proof below. When you are done, have one group member read the sentences aloud one at a time. Do not proceed to the next sentence until every group member can explain why the sentence is true.

**Claim:** $(\forall x \in \mathbf{Q}) x^2 \neq 2$.

**Proof:** We proceed by contradiction. That is, we assume ____________________

In other words, there exist $a, b \in \mathbf{Z}$, so that $x = a/b$ with $a/b$ a fraction in lowest terms, and $2 = x^2 = \frac{a^2}{b^2}$.

Rearranging, we have $a^2 = 2b^2$. Therefore $a^2$ is even. By ________________________, it then follows $a$ is even.

Since $a$ is even, there exists $k \in \mathbf{Z}$ so that ___________. Therefore $a^2 = 4k^2$. Since $a^2 = 2b^2$, it then follows that $b^2 = 2k^2$. Therefore $b^2$ is even.

By Tutorial Question 3.3, it then follows that $b$ is _________. Since $a$ and $b$ are both even, $a/b$ is **not** ___________________________________________. This contradicts our assumption that ________________________________________________.

(b) One can use the method above to prove $\sqrt{p}$ is irrational for any prime number $p$. If you were to do so, what result would you first have to prove? [**Hint**: Trying re-writing the proof above for $\sqrt{3}$ instead of $\sqrt{2}$ and see what needs to change.]

**3.5** (Peirce's Law and Curry's Paradox)**.** We look at a simple identity due to philosopher Charles Peirce (1839–1914). An odd consequence is Curry's Paradox, discovered by logician Haskell Curry (1900–1982). This paradox arises when we allow **self-reference**.

Before we start: *modus ponens* is a rule of deduction that says "if $p \Rightarrow q$ is True and $p$ is True then $q$ is True". (You can check that this is valid by staring at the truth table for $\Rightarrow$.)

You may find modus ponens useful in the following questions.

(a) Using truth tables, show that *Peirce's Law*

$$[(p \Rightarrow q) \Rightarrow p] \Rightarrow p$$

is a tautology.

(b) Suppose that $(p \Rightarrow q) \Leftrightarrow p$ is True. Use part (a) to deduce that $q$ is True.

(c) Argue informally that, for any statement $q$, the self-referential statement

$$p: \text{ "If } p \text{ is True, then } p \text{ implies } q\text{"}$$

satisfies $(p \Rightarrow q) \Leftrightarrow p$.

(d) Use (b) and (c) to conclude that anything is True. What has gone wrong?

**Topic: proofs by mathematical induction**

**3.6** (Proof reading)**.** Consider the following claim and the following proof.

**Claim:** The largest natural number is 1.

*Proof.* Let $N$ be the largest natural number.

Therefore for every $n \in \mathbf{N}$ we have $N \geqslant n$. Since $1 \in \mathbf{N}$, we know $N \geqslant 1$. And since $N^2 \in \mathbf{N}$, we know $N \geqslant N^2$. Therefore $N^2 - N \leqslant 0$. So $N(N-1) \leqslant 0$. Thus $N - 1 \leqslant 0$, meaning that $N \leqslant 1$. We conclude that $N = 1$. □

The claim is False, but all of the logical steps in the proof seem to be reasonable? Explain.

**3.7** (Mathematical induction)**.** Using proofs by induction, prove the following statements:

(a) $(\forall n \in \mathbf{N}) \quad 0^3 + 1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{1}{4}n^2(n+1)^2$;

(b) $n^2 \leqslant n!$ whenever $n \geqslant 4$.

**3.8** (Powers of $i$)**.** Using the fact that $i^2 = -1$ and a proof by induction, show that for all $n \geqslant 0$ we have

$$\begin{aligned} i^{4n+1} &= i \\ i^{4n+2} &= -1 \\ i^{4n+3} &= -i \\ i^{4n+4} &= 1. \end{aligned}$$

**3.9** (Fibonacci numbers)**.** The *Fibonacci sequence* starts with 0 and 1 and proceeds so that each subsequent number is the sum of the previous two:

$$0, 1, 1, 2, 3, 5, 8, 13, 21 \ldots$$

The elements of this sequence are called *Fibonacci numbers*.

The sequence is named after the Italian mathematician Leonardo Bonacci, who included it as an example in his 1202 textbook **Liber Abaci** (**Book of Calculation**). Among other things, the book popularised the use of the Indo-Arabic numeral system in the Western world.

However, Fibonacci was certainly not the first to study the above sequence of numbers. References to it appear as far back as India in 200 BCE, when an Indian author named Pingala used these numbers to count the number of fixed-length music patterns that could be made using short and long syllables. Pingala's work on short and long syllable sequences can be seen as an early form of binary notation. Unfortunately, as is very often the case in mathematics, the name commonly attached to an object is that of a person who rediscovered or popularised it, rather than those who studied it earlier.

For $n \geqslant 0$, let $f_n$ denote the $n$-th Fibonacci number: $f_0 = 0, f_1 = f_2 = 1, f_3 = 2, \ldots$. By definition we have $f_n = f_{n-1} + f_{n-2}$ for all $n \geqslant 2$.

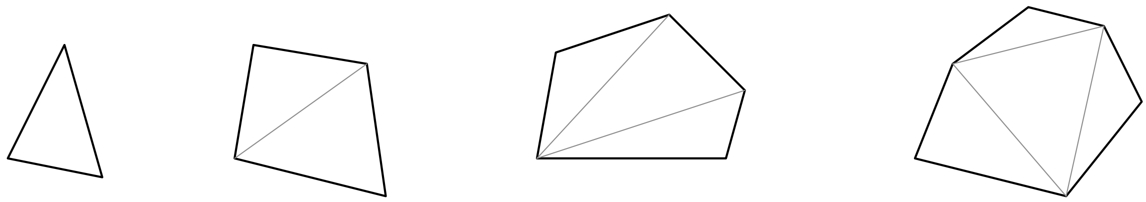
Using a proof by strong induction prove that

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

for all $n \geqslant 0$.

**3.10** (Dividing polygons)**.** Let $n \geqslant 3$ be an integer.

Consider the process of *triangulating* a convex polygon with $n$ sides, that is dividing it fully into triangular regions by means of non-intersecting lines from vertex to non-adjacent vertex. For example here are some triangulated convex polygons with $n = 3, 4, 5, 6$ sides:



(Can't remember what a *convex* polygon is? Google it. . . )

(a) Looking at the above examples (and working out a few more if you need them), guess a formula for the number of regions in a triangulated convex polygon with $n$ sides.

(b) Prove that your formula is correct using a proof by induction. You may use without proof the fact that a single line splits a convex polygon into two convex polygons (with fewer sides).

(c) (highly optional)

Explore what happens if we drop the convexity condition and allow polygons that are concave. First note that not all choices of lines will give you triangulations (some lines may be passing on the outside of the polygon, so they are not actually dividing the polygon).

Does your proof in (b) still work in this setting?

If not, does your formula in (a) still work in this setting?

(Can you see why these two questions could have different answers?)