Topics: set operations, relations

4.1 (Relations on sets)**.** Let $A = \{1, 2, 3\}$ and $B = \{1, 3\}$. Which of the following statements are True? For each, write a sentence that refers to the appropriate definition to justify your response.

(a) $A = B$

(b) $A \subsetneq B$

(c) $B \subseteq A$

(d) $(1, 2) \in A \times B$

(e) $(1, 2) \in B \times A$

(f) $A \cup B$ contains five elements

(g) $A \cup B$ contains three elements

(h) $A \cap B \neq \varnothing$.

4.2 (Quantified statements about sets)**.** Let $A = \{1, 4, 7, 25\}$ and $B = \{2, 7, 60\}$. Which of the following are True?

(a) $(\forall a \in A)\ [(\exists b \in B)\ \ b + a \text{ is even}]$

(b) $(\exists a \in A)\ [(\forall b \in B)\ \ a > b]$.

4.3 (Proofs about sets)**.** Using the definitions of the notations $\subseteq$, $=$, $\cup$, and $\cap$, give a proof of each of the following theorems. To think about how to get started, look back at the proof of Theorems 2.19 and 2.20.

You may use either an informal proof style (with more words and sentences) or a formal one (with mostly symbols and logic notation), but try to keep to one logical step per line, and check that each step follows from the previous ones before moving on.

For the "if and only if" theorems, recall that $p \Leftrightarrow q \equiv [(p \Rightarrow q) \wedge (q \Rightarrow p)]$.

(At first glance some of you may think that the theorems below are **obvious**. If indeed they are **obvious** you should have little trouble writing down a proof that would convince a skeptical peer.)

(a) "Let A, B and C be sets. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$."

Proof. Let A, B and C be sets so that $A \subseteq B$ and $B \subseteq C$. Let $x \in A$.

. . . . (Fill in the missing steps) . . . .

We conclude that $A \subseteq C$. □

(b) "Let A and B be sets. We have $B \subseteq A$ if and only if $A \cap B = B$."

Proof. In one direction, let A and B be sets so that $B \subseteq A$.

. . . . (Fill in the missing steps) . . . .

Therefore $A \cap B = B$.

For the other direction, let A and B be sets so that $A \cap B = B$.

. . . . (Fill in the missing steps) . . . .

Therefore $B \subseteq A$. □

(c) "Let A and B be sets. We have $A \cup B = B$ if and only if $A \subseteq B$."

(Proceed similarly to the previous part.)

(d) Using parts (b) and (c), what can you say about A and B when $A \cap B = A \cup B$?

(e) Give a counterexample to show that the following statement is False in general:

$$A \cup (B \cap C) = (A \cup B) \cap C.$$

4.4 (Relations on sets with two elements)**.** Let $S = \{a, b\}$ be a set with two elements.

(a) Find all possible relations on S.

(b) Which of the above relations are orders on S?

(c) Which of the above relations are functions $S \longrightarrow S$?

4.5 (Ordered tuples)**.** Let A_1 and A_2 be sets. As we have seen in Definition 2.15, an *ordered pair* (a_1, a_2) is a set of the form

$$(a_1, a_2) = \big\{\{a_1\}, \{a_1, a_2\}\big\}, \qquad \text{where } a_1 \in A_1 \text{ and } a_2 \in A_2.$$

(This definition is due to the Polish mathematician Kazimierz Kuratowski.)

(a) Prove that $(a_1, a_2) = (b_1, b_2)$ if and only if $a_1 = b_1$ and $a_2 = b_2$.

(b) Based on the above, give a recursive definition of the n-tuple $(a_1, a_2, \ldots, a_n)$ using ordered pairs. Prove the property

$$(a_1, \ldots, a_n) = (b_1, \ldots, b_n) \;\Leftrightarrow\; a_i = b_i \text{ for all } i = 1, \ldots, n.$$

4.6 (Regularity rules out self-containment)**.** Among the ZFC set axioms we have the Axiom of Regularity: if A is a non-empty set, there exists $y \in A$ such that $A \cap y = \varnothing$.

Use a proof by contradiction to show that there does not exist any set S with the property that $S \in S$.

[**Hint**: Take $A = \{S\}$ in the Axiom of Regularity and find a contradiction.]

Topics: bounds, supremum, infimum, order

4.7 (Finding a supremum)**.** Consider the set

$$A = \left\{1 - \frac{1}{n} : \ n \in \mathbf{N}, n \neq 0\right\}.$$

(a) Draw a picture of A on the real line.

(b) Using your picture, determine for which values of β we can say that β is an upper bound for A in $\mathbf{R}$.

(c) Based on your answer to (b), what do you think is the supremum of A in $\mathbf{R}$?

(d) Suppose that your answer to (c) is incorrect. This means there is an upper bound r that is smaller than the one you have conjectured in (c). Prove that there is an element of A that is bigger than r. At some point in your proof, you might need to use the following version of the Archimedean Principle (see Exercise 2.30):

$$(\forall \varepsilon > 0) \ \ [(\exists n \in \mathbf{N}) \ 1/n < \varepsilon].$$

[**Hint**: It might help to draw a picture. Based on the picture, what do you want to show?]

(e) Does your work in the previous part convince you that your answer in part (c) is correct? Take a moment to refer back to Definition 2.46 to be certain.

4.8 (Inequalities and squares)**.** Let $a, b \in \mathbf{R}_{>0}$. Using the axioms of the order relation on $\mathbf{R}$ and/or results we have proved about this order relation, prove the following:

(a) If $a > b$, then $a^2 > b^2$.

(b) If $a^2 \leqslant b^2$, then $a \leqslant b$.

(c) Give a counterexample to the statement in part (a) if we do not require a and b to be positive.

4.9 (Minima and maxima)**.** Let S be an ordered set and A a subset of S. Consider the following definitions:

- We say that an element $M \in A$ is a *maximum* of A when $(\forall x \in A) \ x \leqslant M$.
- We say that an element $m \in A$ is a *minimum* of A when $(\forall x \in A) \ m \leqslant x$.

(a) Given $M \in A$, prove that M is a maximum of A if and only if (A has a supremum in S and $M = \sup A$).

(There is a similar result for minimum and infimum.)

Find, if they exist, the supremum, infimum, maximum, and minimum of each subset A of $\mathbf{R}$ given below.

Try to justify your answers. You may need to refer to Tutorial Question 4.8 or Theorem 2.60 for some of the parts. You may also assume that π and $\sqrt{3}$ are both irrational.

(b) $A = \{x \in \mathbf{R} : \ x^2 \leqslant 3\}$

(c) $A = \{x \in \mathbf{Q} : \ x^2 \leqslant 3\}$

(d) $A = \{x \in \mathbf{R} : \; x \leqslant \pi\}$

(e) $A = \{x \in \mathbf{Q} : \; x \leqslant \pi\}$.

4.10 (Well-Ordering Property of $\mathbf{N}$)**.** The objective is to prove Theorem 2.21: Every non-empty subset $S \subseteq \mathbf{N}$ has a *minimum*: there exists $m \in S$ such that $m \leqslant x$ for all $x \in S$.

Start by defining the following subset of $\mathbf{N}$:

$$A = \{k \in \mathbf{N} : \text{ every } T \subseteq \mathbf{N} \text{ that contains an element} \leqslant k \text{ has a minimum}\}.$$

(a) Convince yourself (and the person next to you, for good measure) that if we prove $A = \mathbf{N}$, then the statement of the Theorem is proved.

(b) Use induction to prove that $(\forall n \in \mathbf{N})\; n \in A$, therefore $A = \mathbf{N}$.