Topics: real numbers, intervals, inequalities

5.1 (Bounds on intervals)**.** Let a and b be real numbers with $a < b$. For each interval below, determine (if they exist) the maximum, minimum, infimum, and supremum in $\mathbf{R}$.

(Recall the definitions of maximum and minimum from Tutorial Question 4.9.)

Interval	Infimum	Supremum	Minimum	Maximum
(a, b)				
$[a, b]$				
$(a, b]$				
$[a, b)$				
(a, ∞)				
$[a, \infty)$				
$(-\infty, b)$				
$(-\infty, b]$				

5.2 (An interesting real number)**.** For any $n \in \mathbf{N}$, let $f(n)$ be the following sum:

$$f(n) = \sum_{k=0}^{n} \frac{1}{k!}.$$

(Recall that $0! = 1$.)

(a) Compute $f(0), f(1), f(2), f(10), f(11), f(48), f(49)$, and $f(50)$.

[**Hint**: Use Wolfram Alpha to do this quickly: `sum 1/k!  with k from 0 to n`.]

(b) Consider the following set

$$A = \{x \in \mathbf{Q} : \ (\exists n \in \mathbf{N}) \ \ x < f(n)\}.$$

Show that if $x \in A$, then there exists a rational number $y \in A$ such that $x < y$.

(c) Can you guess which number $\sup A$ is? Do you think it is rational or irrational?

(**Note:** We do not yet know how to prove that this set is even bounded above, let alone how to compute its supremum! You may assume the set is bounded above, and make a guess of the supremum based purely on your calculations in Part (a).)

5.3 (Geometry of inequalities)**.** For each inequality, find all values of x that make the inequality True, and illustrate these values on the real line.

(a) $|x| < 4$

(b) $|x + 1| < 4$

(c) $|x - 1| < 4$

(d) $|2x| < 4$

(e) $|x/2| < 4$

(f) $|2x + 10| < 4$

(g) $|-2x + 10| < 4$.

5.4 (Inequalities and transformations)**.** At some point in the past you have learned about geometric transformations (translation, reflection, scaling, etc.) of the plane, and their effects on the graphs of functions.

For example, given a function f we can draw conclusions about the shape of $f(2x + 1) - 3$ using words like **scale** or **translate**.

When you compare your drawings of the solutions in the previous question, perhaps we can see the same sort of behaviour, this time related to geometric transformations of the line.

Let $a, b, c \in \mathbf{R}$, with $a, c > 0$. In answering the following questions, use geometric language such as

(a) What transformation turns the set of solutions for $|x| < c$ into the set of solutions for $|x + b| < c$?

(b) What transformation turns the set of solutions for $|x| < c$ into the set of solutions for $|ax| < c$?

(c) What transformation turns the set of solutions for $|ax + b| < c$ into the set of solutions $|-ax + b| < c$?

(d) What transformation turns the set of solutions for $|ax + b| < c$ into the set of solutions for $|ax + b| \geqslant c$? Can you express this transformation using the language of set theory?

5.5 (Probing the triangle inequality)**.** Give

(a) an example of $x, y \in \mathbf{R}$ such that the triangle inequality is in fact an equality;

(b) an example of $x, y \in \mathbf{R}$ such that the triangle inequality is a strict inequality.

Guided by your examples, complete the following into a True statement: "given $x, y \in \mathbf{R}$, the triangle inequality for $|x + y|$ is an equality if and only if . . . "

5.6 (Neighbourhoods)**.** Let $c, \varepsilon \in \mathbf{R}$ with $\varepsilon > 0$. The *ε-neighbourhood of c* is by definition the interval $(c - \varepsilon, c + \varepsilon)$.

(a) Draw, on the real line, the 0.1-neighbourhood of 2.

(b) Let $x \in \mathbf{R}$. Prove that x is in the ε-neighbourhood of c if and only if $|x - c| < \varepsilon$.

(c) Prove that the ε-neighbourhood of 2 contains a real number other than 2.

[**Hint**: Use the Archimedean Principle.]

Topics: bounds, limits of sequences

5.7 (Bounding functions)**.** To *bound* a function $f : [a,b] \longrightarrow \mathbf{R}$ on the interval $[a,b]$ means to find a real number K such that $|f(x)| \leqslant K$ for all $x \in [a,b]$.

(a) Bound the function $f : [-2,4] \longrightarrow \mathbf{R}$ given by $f(x) = |x-3|$, on $[-2,4]$.

(b) Bound

$$\frac{1}{|x-3|}$$

for $x \in (-2,0)$.

Can it be bounded on $[-2,4] \setminus \{3\} = [-2,3) \cup (3,4]$?

(c) Bound the function $f : (-2,7) \longrightarrow \mathbf{R}$ given by

$$f(x) = 2x^2 - 6x + 7.$$

[**Hint**: Any upper bound on $|f|$ will do! The triangle inequality may be useful here.]

5.8 (Proving a limit (or five))**.**

(a) Look up Definition 3.6 and copy it down or have it handy throughout this question.

(b) Make a guess L about the limit of the sequence

$$(x_n) = \left(\frac{n^2}{3n^2-4}\right).$$

(c) Letting L denote your guess from the previous part, let $p(n,\varepsilon)$ be the condition

$$|x_n - L| < \varepsilon.$$

Find a value $M \in \mathbf{N}$ such that $p(n, 0.01)$ is True for all $n > M$.

(d) Generalising your work in the previous part, find a formula for M as a function of ε so that $p(n,\varepsilon)$ is True for all $n > M$.

(e) Write an informal proof that

$$\lim_{n\to\infty} \frac{n^2}{3n^2-4} = L,$$

where L is your guessed limit. (It's $1/3$, isn't it?)

Your proof should have the following structure:

> Let $\varepsilon > 0$. Let $M =$ (your formula from the previous part).
> Suppose $n > M$.
>
> (Algebraic manipulations as necessary.)
>
> We conclude that $|x_n - L| < \varepsilon$ for all $n > M$.
> As this holds for all $\varepsilon > 0$, it follows that
>
> $$\lim_{n\to\infty} \frac{n^2}{3n^2-4} = L.$$

(f) Was that fun? Then repeat the process in parts (b) to (e) with the sequences

$$(y_n) = \left(\frac{1}{\sqrt{n}} + 1\right)$$
$$(z_n) = \left(\frac{6}{3n^2 - 4}\right)$$
$$(w_n) = \left(\frac{(-1)^n}{2n}\right)$$
$$(u_n) = \big(1, 1, 1, 1, \dots\big).$$

5.9. [Limits are unique]

(a) Suppose $a \in \mathbf{R}$ is such that $|a| < \varepsilon$ for all $\varepsilon > 0$. Then $a = 0$.

(b) Any sequence (x_n) has at most one limit.

5.10. [Limits and inequalities] In Theorem 3.16 we have seen that if (x_n) and (y_n) are convergent and $x_n \leqslant y_n$ for all $n \in \mathbf{N}$, then

$$\lim_{n\to\infty} x_n \leqslant \lim_{n\to\infty} y_n.$$

Suppose now that (x_n) and (y_n) are convergent sequences and $x_n < y_n$ for all $n \in \mathbf{N}$. Does it follow that

$$\lim_{n\to\infty} x_n < \lim_{n\to\infty} y_n?$$

If you think the answer is Yes, give a proof.

If you think the answer is No, give a counterexample.