Topics: convergence of sequences

6.1 (Algebra of Limits: examples)**.** You are told that

$$\left(\frac{1}{n}\right) \longrightarrow 0 \qquad \text{and} \qquad (1,1,1,1,1,\dots) \longrightarrow 1.$$

Apply the Algebra of Limits Theorem and the two facts given above to find the limit of each of the following sequences:

(a) $(x_n) = \left(\dfrac{n^2+6}{3n^2-4}\right)$;

(b) $(y_n) = (3,3,3,3,\dots)$.

6.2 (Comparisons are your Friend)**.** Familiarise yourself with the statement of the Sandwich Theorem, see Exercise 3.7.

(a) Find an inequality relation between n^2 and $n^2+6n+11$ that is valid for all $n \in \mathbf{N}$. Use this to find

$$\lim_{n\to\infty} \frac{1}{n^2+6n+11}.$$

(b) Modify the approach in part (a) appropriately so you can find

$$\lim_{n\to\infty} \frac{1}{n^2-6n+4}.$$

(c) Find the limit of the sequence (x_n) given by

$$(x_n)_{n\geqslant 1} = \left(\frac{\sin(n)}{n}\right)_{n\geqslant 1}.$$

6.3 (A more intricate limit)**.** The final objective of this question is to determine (with proof) the limit of the sequence

$$(x_n)_{n\geqslant 1} = \left(n^{1/n}\right)_{n\geqslant 1}.$$

(a) Let (y_n) be a sequence and suppose that $y_n^2 \longrightarrow 0$. Prove that (y_n) converges and $y_n \longrightarrow 0$.

Would your argument work if we had instead $y_n^2 \longrightarrow 1$?

(b) From now on we consider the sequence (x_n) defined at the beginning of the question.

Prove that $1 \leqslant x_n$ for all $n \geqslant 1$.

[**Hint**: First use induction to show that if t is a real number with $0 < t < 1$, then $0 < t^n < 1$ for all $n \geqslant 1$.]

(c) For any $n \geqslant 1$, let $y_n = x_n - 1$, so that $x_n = 1 + y_n$.

Use the Binomial Formula

$$(a+b)^n = \sum_{j=0}^{n} \binom{n}{j} a^{n-j} b^j \qquad \text{for all } a, b \in \mathbf{R}, n \in \mathbf{N}, n \geqslant 1,$$

to show that $2 > (n-1)y_n^2$ for all $n \geqslant 2$.

[**Hint**: Apply the Binomial Formula to $(1+y_n)^n$ and pick out the term corresponding to $j = 2$.]

(d) Use the Sandwich Theorem to find the limit of (y_n^2) as $n \longrightarrow \infty$.

(e) Find the limit of (x_n) as $n \longrightarrow \infty$.

6.4 (Divergent Sequences)**.** Recall the precise statement of Theorem 3.22.

For each sequence below, use the Theorem to determine if the sequence diverges. Caution! There may be some sequences below for which the Theorem gives us no information.

(a) $(a_n) = (\sqrt{n})$;

(b) (b_n) given by

$$b_n = \begin{cases} 1 & n = 3k \\ \frac{1}{n} & n = 3k+1 \\ -\frac{1}{2n} & n = 3k+2; \end{cases}$$

(c) (c_n) given by

$$c_n = \begin{cases} 0 & n = 3k \\ \frac{1}{n} & n = 3k+1 \\ -\frac{1}{2n} & n = 3k+2; \end{cases}$$

(d) $(d_n) = (n - 10^{100}\sqrt{n})$.

Topics: Cauchy sequences and applications

6.5 (Set(s) of Sequences)**.** Draw a Venn-style diagram that illustrates the set Seq of all sequences (of real numbers). On this diagram, mark:

- the subset Bd of all bounded sequences;
- the subset Mon of all monotone sequences;
- the subset Cau of all Cauchy sequences;
- the subset Conv of all convergent sequences;
- the subset Div of all divergent sequences.

To do this accurately, you need to ask yourself: for any pair (A, B) of the above subsets, how are A and B related (e.g. are they disjoint? Is one a subset of the other? Are there elements in one that are not in the other?)

See if you can find explicit examples of sequences that belong to the various regions appearing in your diagram.

6.6 (A Recursive Sequence)**.** Let (a_n) be the sequence defined by $a_0 = 0$ and

$$a_{n+1} = \sqrt{2 + a_n} \qquad \text{for all } n \geqslant 0.$$

(a) Compute the first few terms of the sequence.

(b) Use induction to prove that $0 \leqslant a_n < 2$ for all $n \in \mathbf{N}$.

(c) You now know that (a_n) is bounded. What other property would you need to prove for (a_n) so that you can conclude that (a_n) converges?

(d) Use induction to prove the property you identified in the previous part, and therefore conclude that (a_n) converges.

(e) What is the limit of the sequence (a_n)?

6.7 (A Contractive Sequence)**.** Let $(a_n)_{n\geqslant 1}$ be the sequence defined by $a_1 = 1$, $a_2 = 2$, and

$$a_n = \frac{a_{n-2} + a_{n-1}}{2} \qquad \text{for all } n \geqslant 3.$$

(a) Compute the first few terms of the sequence.

(b) Use induction to prove that $1 \leqslant a_n \leqslant 2$ for all $n \geqslant 1$.

(c) Is the sequence monotone? Explain.

(d) Prove that for all $n \geqslant 1$ we have

$$|a_{n+2} - a_{n+1}| = \frac{1}{2}\,|a_{n+1} - a_n|.$$

(e) Conclude that the sequence converges. Can you find the limit?

6.8 (Squares, Squares Everywhere)**.** Consider the following statement:

(*) For any $m, y \in \mathbf{Q}$ such that $0 < m^2 < y$, there exists $t \in \mathbf{Q}$ such that $m^2 < t^2 < y$.

(a) Prove the statement. Aim for a short and simple proof, involving properties of the real numbers that we have seen.

[**Hint**: Use Theorem 2.60.]

Read the statement (*) again and note that it does not involve the real numbers $\mathbf{R}$ at all. Is it possible to give a proof of statement (*) that does not involve $\mathbf{R}$ at all, only properties of the rational numbers $\mathbf{Q}$? Let's see.

(b) Prove the Archimedean Property of $\mathbf{N}$ in $\mathbf{Q}$: for every $x \in \mathbf{Q}$ with $x > 0$ there exists $n \in \mathbf{N}$ such that $n > x$.

(c) Prove statement (*) without any reference to $\mathbf{R}$.

[**Hint**: Consider $t = m + \frac{1}{k}$, where $k \in \mathbf{N}$ is chosen such that $(2m+1) < k(y - m^2)$.]

(d) Is there a statement (*) that replaces squares by cubes, fourth powers, etc?

Write down a precise formulation of statement (*) for n-th powers (with $n \in \mathbf{N}$, $n \geqslant 1$). Modify your proofs in parts (a) and (c) so that they work in this more general setting.