Topics: more sequences; limit points

7.1 (Zeno's Non-Paradox)**.** For $n \in \mathbf{N}$, let

$$x_n = \sum_{k=0}^{n} \frac{1}{2^k}.$$

(a) Write out the first few terms of the sequence (x_n).

(b) Prove that (x_n) is bounded and monotone.

(c) Prove that $\sup\{x_n : \ n \in \mathbf{N}\} = 2$.

(d) Find the limit of (x_n) as $n \longrightarrow \infty$.

7.2 (Alternating Divergent)**.** Let (x_n) be a sequence converging to a limit $L \in \mathbf{R}$ and let (y_n) be the sequence given by $y_n = (-1)^n x_n$ for all $n \in \mathbf{N}$.

(a) Prove that if $L \neq 0$ then (y_n) diverges.

(b) What happens if $L = 0$? Make a precise statement and prove it.

7.3 (Limit Points of $\mathbf{N}$)**.** In Example 4.5 we have seen that if $a \in \mathbf{N}$, then a is not a limit point of $\mathbf{N} \subseteq \mathbf{R}$.

Now prove that if $a \in \mathbf{R} \setminus \mathbf{N}$, then a is not a limit point of $\mathbf{N} \subseteq \mathbf{R}$.

7.4 (Limit Points)**.** Find all limit points of each of the following subsets of $\mathbf{R}$:

(a) $(1, 2)$

(b) $(1, 2]$

(c) $\{0\}$

(d) $\mathbf{Q}$

(e) $(-\infty, -1) \cup (1, \infty)$

(f) $\mathbf{R} \setminus \mathbf{N}$

(g) $\{1/n : \ n \in \mathbf{N}\}$.

7.5 (Limits from Scratch)**.** Using the $\varepsilon - \delta$ definition of limit of a function, prove that each of the statements below is True.

(a) $\lim_{x \to 2} 1 = 1$

(b) $\lim_{x \to a} 1 = 1$

(c) $\lim_{x \to 2} x = 2$

(d) $\lim_{x \to a} x = a$

(e) $\lim_{x \to 2} x^2 = 4$

(f) $\lim_{x \to a} x^2 = a^2$ for $a > 0$.

Topics: limits of functions, one-sided limits, sequential methods

7.6 (One-Sided Limits)**.** Let $E \subseteq \mathbf{R}$, let $f : E \longrightarrow \mathbf{R}$ be a function, and let $a, L \in \mathbf{R}$.
Suppose that a is a limit point of the set $E \cap (a, \infty)$.
We write

$$\lim_{x \to a^+} f(x) = L$$

if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $x \in E$ and $a < x < a + \delta$, then $|f(x) - L| < \varepsilon$.

(a) Write down the analogous definition of $\lim_{x \to a^-} f(x) = L$.

(b) Consider the function $f : \mathbf{R} \longrightarrow \mathbf{R}$ given by

$$f(x) = \begin{cases} 2x & x > 1, \\ -2x & x < 1, \\ 0 & x = 1. \end{cases}$$

Sketch a graph of the function.

Prove that

$$\lim_{x \to 1^-} f(x) = -2 \qquad \text{and} \qquad \lim_{x \to 1^+} f(x) = 2.$$

(c) Back to the general setting.

Let $E \subseteq \mathbf{R}$, $f : E \longrightarrow \mathbf{R}$ a function, and $L \in \mathbf{R}$. Suppose that $a \in \mathbf{R}$ is a limit point of both sets $E \cap (-\infty, a)$ and $E \cap (a, \infty)$. Prove that

$$\lim_{x \to a} f(x) = L \quad \text{if and only if} \quad \lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x).$$

(d) What can you say about the limit of the function f from part (b) as $x \longrightarrow 1$?

7.7 (Inequalities and Limits of Functions)**.** Prove Theorem 4.12:
"Let $E \subseteq \mathbf{R}$, a a limit point of E, and $f, g : E \longrightarrow \mathbf{R}$ such that

$$\lim_{x \to a} f(x) = \alpha \quad \text{and} \quad \lim_{x \to a} g(x) = \beta.$$

If $f(x) \leqslant g(x)$ for all $x \in E$, then $\alpha \leqslant \beta$."
[**Hint**: Combine Theorem 4.10 and Theorem 3.16.]

7.8. Prove the Sandwich Theorem for Functions, which is the following statement:

Let $E \subseteq \mathbf{R}$, let a be a limit point of E, let $f, g, h : E \longrightarrow \mathbf{R}$, and $L \in \mathbf{R}$. Suppose

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

and

$$f(x) \leqslant g(x) \leqslant h(x) \quad \text{for all } x \in E.$$

Then

$$\lim_{x \to a} g(x) = L.$$

7.9. Using the Sandwich Theorem, find

$$\lim_{x \to 0} x^2 \sin(x).$$

7.10 (Characteristic Function of $\mathbf{Q}$)**.** Let $g : \mathbf{R} \longrightarrow \mathbf{R}$ be defined by

$$g(x) = \begin{cases} 1 & x \in \mathbf{Q} \\ 0 & x \notin \mathbf{Q}. \end{cases}$$

Let $a \in \mathbf{R}$.

(a) Prove that there exists a sequence (a_n) with $a_n \in \mathbf{Q}$ for all $n \in \mathbf{N}$ and $a_n \longrightarrow a$.

[**Hint**: Use the fact that $\mathbf{Q}$ is dense in $\mathbf{R}$, see Corollary 2.61.]

(b) It is also the case that $\mathbf{R} \setminus \mathbf{Q}$ is dense in $\mathbf{R}$. Use this to prove that there exists a sequence (b_n) with $b_n \in \mathbf{R} \setminus \mathbf{Q}$ for all $n \in \mathbf{N}$ and $b_n \longrightarrow a$.

(c) Find $\lim_{x \to a} g(x)$ (or show that it does not exist).