Topics: definition and properties of continuity

8.1 (Proving and Disproving Continuity).

(a) Using the definition of continuity, prove that $f: \mathbf{R} \longrightarrow \mathbf{R}$ given by $f(x) = 3x + 1$ is continuous at 0.

(b) Fix $k \in \mathbf{R}$. Using the definition of continuity, prove that the constant function $f: \mathbf{R} \longrightarrow \mathbf{R}$ given by $f(x) = k$ is continuous.

(c) Let $f, g: \mathbf{R} \longrightarrow \mathbf{R}$ be continuous at c. Using the definition of continuity, prove that $2f + 2g$ is continuous at c.

(d) Carefully write the negation of the definition of continuity: "The function $f: E \longrightarrow \mathbf{R}$ is not continuous at $a \in E$ if ..."

(e) Using part (d), show that the following function $f: \mathbf{R} \longrightarrow \mathbf{R}$ is not continuous at 0:

$$f(x) = \begin{cases} |x| & x \neq 0, \\ 1 & x = 0. \end{cases}$$

8.2 (Continuity and Local Boundedness). Let $f: \mathbf{R} \longrightarrow \mathbf{R}$ be continuous at c. Using the definition of continuity, prove the following facts:

(a) There exists $\delta_+ > 0$ so that $f(x)$ is bounded above by $f(c) + \frac{1}{2}$ on $(c - \delta_+, c + \delta_+)$.

(b) There exists $\delta_- > 0$ so that $f(x)$ is bounded below by $f(c) - \frac{1}{2}$ on $(c - \delta_-, c + \delta_-)$.

(c) For every $\varepsilon > 0$ there exists $\delta > 0$ so that

$$f(c) - \varepsilon < f(x) < f(c) + \varepsilon \qquad \text{for all } x \in (c - \delta, c + \delta).$$

8.3 (Sequences are Continuous).

(a) Let $E \subseteq \mathbf{R}$, $f: E \longrightarrow \mathbf{R}$, and $a \in E$. Prove that if a is **not** a limit point of E, then f is continuous at a.

From now on, fix a sequence (f_n), and think of it as a function $f: \mathbf{N} \longrightarrow \mathbf{R}$ given by $f(n) = f_n$.

(b) What are the limit points of $\mathbf{N}$?

(c) Is f a continuous function on $\mathbf{N}$?

(d) Consider the statement: "every sequence is a continuous function $\mathbf{N} \longrightarrow \mathbf{R}$". Based on the work you have done above, is this statement True?

8.4 (Restriction and Continuity). Let $E \subseteq \mathbf{R}$ and let $f: E \longrightarrow \mathbf{R}$.

Given a subset $A \subseteq E$, recall that the *restriction of f to A* is the function $f|_A: A \longrightarrow \mathbf{R}$ given by

$$f|_A(x) = f(x) \qquad \text{for all } x \in A.$$

Prove that if $a \in A$ and f is continuous at a, then $f|_A$ is continuous at a.

Topics: extreme and intermediate value theorems

Refresh your memory on the precise statements of the Extreme Value Theorem and the Intermediate Value Theorem.

8.5 ("Counterexamples")**.**

(a) Give an example of a continuous function $f:\ E \longrightarrow \mathbf{R}$ such that: for every $x \in E$ there exists $x' \in E$ such that $f(x) < f(x')$.

(b) Why does the Extreme Value Theorem not apply to your function from (a)?

(c) Give an example of a function $f:\ E \longrightarrow \mathbf{R}$ such that: there exist $a, b \in E$ and $y \in \mathbf{R}$ such that

$$f(a) < y < f(b),$$

but for all $c \in [a, b]$ we have $f(c) \neq y$.

(d) Why does the Intermediate Value Theorem not apply to your function from (c)?

8.6. Let $h:\ [-2, 2] \longrightarrow \mathbf{R}$ be given by $h(x) = x^3 - 2x^2 + 1$. Let S be defined as in the proof of Lemma 4.33 to the Intermediate Value Theorem, namely

$$S = \{x \in [-2, 2]:\ h(t) \leqslant 0 \text{ for all } t \in [-2, x]\}.$$

What is $\sup S$?

[**Hint**: Find all the solutions of the equation $h(x) = 0$.]

8.7. Use the definition of continuity to prove that: if $E \subseteq \mathbf{R}$, c is a limit point of E, $f:\ E \longrightarrow \mathbf{R}$ is continuous at c, and $f(c) < 0$, then there exists $\delta > 0$ such that

$$f(x) < 0 \qquad \text{for all } x \in (c - \delta, c + \delta).$$

8.8. Use the Intermediate Value Theorem in the following. Make sure to check that the hypotheses of the Theorem hold before applying it. (You may use without proof the fact that $\sin x$ is continuous on $\mathbf{R}$ and $\sqrt{x}$ is continuous on $[0, \infty)$.)

(a) Prove that the equation $x^2 = 12 + \sqrt{x}$ has at least one real solution.

(b) Prove that the equation $2 \sin x = x$ has at least three real solutions.

(c) Prove that the equation $x^3 = 2x^2 + 3x - 2$ has at least three real solutions.

8.9. Use the Intermediate Value Theorem in the following. Make sure to check that the hypotheses of the Theorem hold before applying it.

(a) Let $f, g:\ [a, b] \longrightarrow \mathbf{R}$ be continuous such that

$$f(a) < g(b) < g(a) < f(b).$$

Prove that there exists $c \in [a, b]$ such that $f(c) = g(c)$.

(b) Let $f:\ [0, 1] \longrightarrow [0, 1]$ be continuous. Prove that there exists $c \in [0, 1]$ such that $f(c) = c$. (In other words, f has at least one *fixed point* on $[0, 1]$.)

(c) Can you find explicit functions f as in part (b) that have: exactly one fixed point? infinitely many fixed points? exactly three fixed points?

[**Hint**: Draw some pictures first, then see if you can find functions that match your pictures.]