Topics: pointwise and uniform convergence

9.1 (Pointwise convergence)**.** For each of the following sequences of functions $f_n : \mathbf{R} \longrightarrow \mathbf{R}$, decide if it converges pointwise on the domain $\mathbf{R}$ to some function f, and if so, find f.

(a) $f_n(x) = \frac{x^2}{n}$ for $n \geqslant 1$;

(b) $f_n(x) = \sin(nx^2)$ for $n \geqslant 0$;

(c) $f_n(x) = \frac{1}{nx^2+1}$ for $n \geqslant 0$.

9.2 (Pointwise but not uniform)**.** The objective is to prove, using the definitions, that the sequence of functions $f_n : [0,1] \longrightarrow \mathbf{R}$ defined by $f_n(x) = x^n$ converges pointwise, but not uniformly, to the function

$$f(x) = \begin{cases} 0 & 0 \leqslant x < 1 \\ 1 & x = 1. \end{cases}$$

(a) Using any graphing calculator or system (e.g. Wolfram Alpha is available online and will work), graph some example functions $y = x^n$, $0 \leqslant x \leqslant 1$. (Say, for $n = 1,\ 2,\ 10,\ 20.$) What do you notice?

(b) Now we prove the sequence converges pointwise:

i. Show that the sequence (x^n) is monotone decreasing for all $0 \leqslant x < 1$.

ii. Show that, for each $0 \leqslant x < 1$, 0 is the greatest lower bound of $\{x^n | n \in \mathbf{N}\}$.

iii. Using the above, show that $f_n \longrightarrow f$ pointwise on $[0,1]$.

(c) Finally, we prove that the convergence is not uniform:

i. Write out the statement "(f_n) does not converge uniformly to f" in formal mathematical language.

ii. Show that, for every $n \in \mathbf{N}$, there exists a real number $c \in (0,1)$ such that $c^n \geqslant 1/2$.

iii. Now prove that the sequence (f_n) does not converge uniformly to f.

9.3 (Uniformly, but where?)**.** Uniform convergence is a property of a sequence of functions on a given domain. We explore what this means in an example.

(a) Carefully write a proof that the sequence of functions $f_n(x) = x/n$ converges pointwise to 0 on the domain $\mathbf{R}$. Pay attention to your choice of M and how it depends on x.

(b) Show that the sequence $f_n(x) = x/n$ **does not** converge uniformly on the domain $\mathbf{R}$.

(c) Show that, in contrast, the sequence $f_n(x) = x/n$ **does** converge uniformly on the domain $[-1,1]$.

(d) What happens if you replace $f_n(x) = x/n$ by $g_n(x) = \frac{1}{nx^2+1}$ above? Is part (c) still True? Briefly explain.

(e) Complete the following statement, and prove it:

Let $E \subseteq \mathbf{R}$. Then the sequence of functions $f_n(x) = x/n$ converges uniformly on E if and only if [. . .].

Topics: definition and properties of derivatives

9.4 (A differentiable piecewise function)**.** Let $k \in \mathbf{R}$, and let $f: \mathbf{R} \longrightarrow \mathbf{R}$ be defined by

$$f(x) = \begin{cases} kx - k + 1 & x < 1, \\ x^3 & x \geqslant 1. \end{cases}$$

Determine all values of k for which f is differentiable at $c = 1$.

9.5 (Derivatives of increasing functions)**.** Let $f: [a,b] \longrightarrow \mathbf{R}$ be a function.
We say that f is *monotone increasing on $[a,b]$* if for all $x, z \in [a,b]$ with $x \leqslant z$ we have $f(x) \leqslant f(z)$.
Prove that if f is differentiable and monotone increasing on $[a,b]$, then $f'(c) \geqslant 0$ for all $c \in [a,b]$.

9.6 (Locally non-constant)**.** Let $f: [a,b] \longrightarrow \mathbf{R}$ be differentiable on $[a,b]$ and let $c \in (a,b)$.

(a) Prove that if $f'(c) \neq 0$, then there exists $\delta > 0$ so that $f(x) \neq f(c)$ for all $x \in (c-\delta, c+\delta)$ with $x \neq c$.

(b) Is the converse of the statement in (a) True?

9.7 (A discontinuous derivative)**.** Consider the function $f: \mathbf{R} \longrightarrow \mathbf{R}$ given by

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & x \neq 0, \\ 0 & x = 0. \end{cases}$$

(a) Determine whether f is continuous on $\mathbf{R}$.

(b) Find the derivative at all points $c \neq 0$.

(c) Find the derivative at $c = 0$. Appeal to left and right side limits.

(d) Determine whether $f'(x)$ is continuous at 0.

9.8 (Smoothing out the corner)**.** Consider the function $g: \mathbf{R} \longrightarrow \mathbf{R}$ given by $g(x) = |x|$.
We have seen in Example 5.3 that g is not differentiable at 0.

(a) Fix $n \geqslant 1$. Find real numbers $a_n, b_n > 0$ such that the function $g_n: \mathbf{R} \longrightarrow \mathbf{R}$ defined by

$$g_n(x) = \begin{cases} x^{2n} + b_n & \text{if } |x| < a_n \\ |x| & \text{otherwise} \end{cases}$$

is continuous and differentiable on $\mathbf{R}$.

(b) Try to sketch the graphs of g_1, g_2, g_3, and $|x|$ on the same graph (say between $x = -2$ and $x = 2$). What do you think the behaviour of the sequence (g_n) is as $n \longrightarrow \infty$? Can you guess what the pointwise limit function is?