Topics: Mean Value Theorem

10.1 (Exploring the statement of Rolle's Theorem)**.** We explore why each of the hypotheses in Rolle's Theorem is required. To this end, in each of the following scenarios, draw a picture of a function that satisfies the given conditions, but has $f'(c) \neq 0$ for all $c \in (a, b)$.

(a) f is continuous on $[a, b]$, differentiable on (a, b), but $f(a) \neq f(b)$.

(b) f is continuous on $[a, b]$, has $f(a) = f(b) = 0$, but is not differentiable on (a, b).

(c) f is differentiable on (a, b), has $f(a) = f(b) = 0$, but f is not continuous on $[a, b]$.

10.2 (Exploring the statement of the Mean Value Theorem)**.** Let f be continuous on $[a, b]$ and differentiable on (a, b). For each of the criteria below, draw a picture of f satisfying it. If such f does not exist, briefly explain why not.

(a) For every $c \in (a, b)$ we have $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

(b) There exists $c \in (a, b)$ so that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

(c) For every $c \in (a, b)$ we have $f'(c) > \dfrac{f(b) - f(a)}{b - a}$.

(d) There exists $c \in (a, b)$ so that $f'(c) > \dfrac{f(b) - f(a)}{b - a}$.

10.3 (Exploring the proof of the Mean Value Theorem)**.** Let f be continuous on $[a, b]$ and differentiable on (a, b). Recall the definition of s in the proof of the Mean Value Theorem:

$$s(x) = \frac{f(b) - f(a)}{b - a}(x - a) + f(a).$$

Prove that if $s(x) = 0$ for all $x \in [a, b]$, then there exists $c \in (a, b)$ so that $f'(c) = 0$.
[**Hint**: Rolle's Theorem.]

10.4 (Use the Mean Value Theorem)**.** Let f be differentiable on $[a, b]$ and let $M \in \mathbf{R}$. Prove that if $f'(x) \geqslant M$ for all $x \in [a, b]$, then

$$M \leqslant \frac{f(b) - f(a)}{b - a}.$$

10.5 (Contractive sequence)**.** Consider the sequence given by $x_0 = \frac{3}{2}$ and

$$x_{n+1} = -\frac{x_n^3}{12} + x_n + \frac{1}{4} \qquad \text{for all } n \in \mathbf{N}.$$

Let $f : \mathbf{R} \longrightarrow \mathbf{R}$ be given by the formula

$$f(x) = -\frac{x^3}{12} + x + \frac{1}{4},$$

so that $x_{n+1} = f(x_n)$ for all $n \in \mathbf{N}$.

(a) Find the global minimum and global maximum of f on the interval $[1, 2]$, and deduce that $1 \leqslant x_n \leqslant 2$ for all $n \in \mathbf{N}$.

(b) Use the Mean Value Theorem for f on the interval $[x_n, x_{n+1}]$, and deduce that (x_n) is a contractive sequence.

(c) Find the limit of (x_n).

Topics: Riemann integrals and Fundamental Theorem of Calculus

10.6 (Partitions and Riemann sums). Consider $f: [-3,2] \longrightarrow \mathbf{R}$ with $f(x) = 4 - x^2$ and the partition $P = \left\{-3, -1, \frac{1}{2}, 2\right\}$.

(a) Find the lower sum $L(f,P)$.

(b) Find the upper sum $U(f,P)$.

(c) Using calculus knowledge, compute $\displaystyle\int_{-3}^{2} f(x)\,dx$ and compare it to the upper and lower sums. Does the result make sense?

10.7 (Integrating a constant function). Let $k \in \mathbf{R}$ and let $f: [a,b] \longrightarrow \mathbf{R}$ with $f(x) = k$. Let $P = \left\{x_0, x_1, x_2, \ldots, x_n\right\}$ be a partition of $[a,b]$.

(a) Find expressions for the lower sum $L(f,P)$ and the upper sum $U(f,P)$.

(b) Find $L(f)$.

(c) Find $U(f)$.

(d) Determine if f is integrable. If it is integrable, find $\displaystyle\int_a^b f(x)\,dx$.

10.8 (Bounding Riemann integrals). Let $f: [a,b] \longrightarrow \mathbf{R}$ be a function and let

$$A_f = \{f(x) : \ x \in [a,b]\}.$$

be the image of f. Suppose that f is bounded, in other words that the set A_f is bounded.

(a) Prove that

$$(b-a) \inf A_f \leqslant L(f).$$

(b) What do you think the correct analogous statement for $U(f)$ is?

10.9 (One point does not matter I). Let $c \in [a,b]$ and consider the function $h: [a,b] \longrightarrow \mathbf{R}$ defined by

$$h(x) = \begin{cases} 0 & \text{if } x \neq c, \\ 1 & \text{if } x = c. \end{cases}$$

Prove that h is integrable and $\displaystyle\int_a^b h(x)\,dx = 0$.

[**Hint**: Use Theorem 5.29 to prove integrability, then determine $L(f,P)$ for an arbitrary partition P of $[a,b]$.]

10.10 (One point does not matter II). Let $f: [a,b] \longrightarrow \mathbf{R}$ be an integrable function and let $c \in [a,b]$. Let $r \in \mathbf{R}$ and consider the function $g: [a,b] \longrightarrow \mathbf{R}$ defined by

$$g(x) = \begin{cases} f(x) & \text{if } x \neq c, \\ r & \text{if } x = c. \end{cases}$$

(a) Use Tutorial Question 10.9 and Theorem 5.31 to show that g is integrable on $[a,b]$ and

$$\int_a^b g(x)\,dx = \int_a^b f(x)\,dx.$$

(b) Consider the slogan "Finitely many points do not matter". Turn it into a precise mathematical statement generalising part (b), then prove the statement.