

Topics: Improper Integrals

11.1 (Integration on Unbounded Intervals I). Consider the function $g : [0, \infty) \rightarrow \mathbf{R}$ given by $g(u) = u^2$. Then g is integrable on $[0, x]$ for all $x > 0$. Define $G : (0, \infty) \rightarrow \mathbf{R}$ by

$$G(x) = \int_0^x u^2 du.$$

- (a) Make an educated guess about the behaviour of G as $x \rightarrow \infty$. Draw a picture to illustrate this.
- (b) Prove that your guess from part (a) is correct.

11.2 (Integration on Unbounded Intervals II). Consider the function $g : [0, \infty) \rightarrow \mathbf{R}$ given by

$$g(u) = \frac{1}{1 + u^2}.$$

Since g is continuous, it is integrable on $[0, x]$ for all $x > 0$. Define $G : (0, \infty) \rightarrow \mathbf{R}$ by

$$G(x) = \int_0^x \frac{1}{1 + u^2} du.$$

- (a) Use your knowledge of integration to find a formula for G that does not involve any integrals.
- (b) Find $L \in \mathbf{R}$ such that

$$\lim_{x \rightarrow \infty} G(x) = L.$$

You may use facts that you know from calculus here. (But you can also think about how you would go about proving things rigorously.)

- (c) Interpreting the integral as an area under the curve, draw a picture and think about what the result of part (b) means.

11.3 (Integration on Unbounded Intervals III). Let $g : [a, \infty]$ be integrable on $[a, b]$ for all $b > a$. We define

$$\int_a^\infty g(u) du = \lim_{x \rightarrow \infty} \int_a^x g(u) du$$

and we call it an *improper integral on an unbounded interval of integration*. We say that the improper integral *converges* if the limit on the right hand side converges.

For each integral below determine, without proof, whether the improper integral converges or diverges:

- (a) $\int_1^\infty u^2 du$;
- (b) $\int_1^\infty \frac{1}{u^2} du$;
- (c) $\int_1^\infty \frac{\log u}{u^2} du$.

[**Hint:** In (c), you may want to use integration by parts to find an antiderivative first.]

11.4 (Integration of Unbounded Integrands I). Consider the function $g : (0, 1] \rightarrow \mathbf{R}$ given by

$$g(u) = \frac{1}{\sqrt{u}}.$$

- (a) Prove that g is unbounded on $(0, 1]$.

So our definition of integrable does not apply to g on $(0, 1]$.

However, g is integrable on $[x, 1]$ for all x such that $0 < x < 1$. Define $G : (0, 1) \rightarrow \mathbf{R}$ by

$$G(x) = \int_x^1 \frac{1}{\sqrt{u}} du.$$

- (b) Using your knowledge of integration, find a formula for $G(x)$ that does not involve any integrals.
- (c) Find $L \in \mathbf{R}$ such that

$$\lim_{x \rightarrow 0^+} G(x) = L.$$

- (d) Interpreting the integral as an area under the curve, draw a picture and think about what the result of part (b) means.

11.5 (Integration of Unbounded Integrands II). Let $g : (a, b] \rightarrow \mathbf{R}$ be integrable on $[c, b]$ for all $c \in (a, b)$. We define

$$\int_a^b g(u) du = \lim_{x \rightarrow a^+} \int_x^b g(u) du$$

and call it an *improper integral of an unbounded integrand*. We say that the improper integral *converges* if the limit on the right hand side converges.

For each integral below determine, without proof, whether the improper integral converges or diverges:

(a) $\int_0^1 \frac{1}{u^2} du;$

(b) $\int_0^1 \frac{1}{\sqrt{1-u^2}} du;$

(c) $\int_{1/3}^1 \frac{1}{3u-1} du.$

Topics: Series

11.6 (Apply Convergence Tests). Using appropriate convergence tests determine whether each of the following series converge.

(a) $\sum_{k=1}^{\infty} 2^k$

(e) $\sum_{k=0}^{\infty} \frac{5^{k+4}}{2^{2k}}$

(b) $\sum_{k=1}^{\infty} 9^{-k}$

(f) $\sum_{k=1}^{\infty} \sin k$

(c) $\sum_{k=1}^{\infty} \frac{2^k}{k!}$

(g) $\sum_{k=1}^{\infty} \frac{1}{k^{\sqrt{2}}}$

(d) $\sum_{k=1}^{\infty} (-1)^k \frac{3}{k}$

(h) $1 - \frac{1}{3} + \frac{1}{9} - \frac{1}{27} + \dots$

11.7 (Telescoping Series). Let (a_k) be a sequence and consider the series

$$\sum_{k=0}^{\infty} (a_k - a_{k+1})$$

Prove that if $a_k \rightarrow 0$, then the series converges to a_0 .

11.8 (Where is this convergent?). Let t be a fixed real number and consider the series

$$1 + \frac{1}{3}t + \frac{1}{9}t^2 + \frac{1}{27}t^3 + \dots$$

- (a) Check that this is a geometric series by finding r , the common ratio between successive terms in the series.
- (b) Notice that your result from the last question is a function of t . Determine all values of t for which $|r| < 1$.
- (c) Using the Geometric Series test, prove the sequence converges if and only if t is among the values you found in the previous part.

11.9 (Comparison Tests). Let (a_k) and (b_k) be sequences.

- (a) Suppose that

$$0 \leq a_k \leq b_k \quad \text{for all } k \in \mathbf{N}.$$

Prove that if $\sum_{k=0}^{\infty} b_k$ converges, then $\sum_{k=0}^{\infty} a_k$ converges.

What can you say if $\sum_{k=0}^{\infty} a_k$ diverges?

- (b) Suppose that there exists $M \in \mathbf{N}$ such that

$$0 \leq a_k \leq b_k \quad \text{for all } k > M.$$

Show that the conclusions of part (a) still apply.

(c) Suppose a_k and b_k are positive and

$$\lim_{k \rightarrow \infty} \frac{a_k}{b_k} \neq 0.$$

Prove that $\sum_{k=0}^{\infty} a_k$ converges if and only if $\sum_{k=0}^{\infty} b_k$ converges.

11.10 (Alternating but not Monotone). Consider the alternating series

$$\sum_{k=0}^{\infty} (-1)^k b_k = \frac{1}{\sqrt{2}-1} - \frac{1}{\sqrt{2}+1} + \frac{1}{\sqrt{3}-1} - \frac{1}{\sqrt{3}+1} + \frac{1}{\sqrt{4}-1} - \dots$$

- (a) Write down formulas for the even terms b_{2k} and for the odd terms b_{2k+1} .
- (b) Show that $b_k \rightarrow 0$ as $k \rightarrow \infty$.
- (c) Show that b_k is not monotone decreasing.

(d) Relate the partial sum A_{2n-1} to the harmonic series and conclude that $\sum_{k=0}^{\infty} (-1)^k b_k$ diverges.