**Topics: connectives, truth tables, tautology, contradiction, equivalence**

Take a moment to write down the truth tables for the following connectives:

- negation $\neg p$
- conjunction $p \wedge q$
- disjunction $p \vee q$
- implication $p \Rightarrow q$
- biconditional $p \Leftrightarrow q$.

**2.1** (Truth tables)**.** Using truth tables, for each of the compound statements below determine all possible truth values of $p$ and $q$ that make the statement True.

(a) $[p \wedge (p \Rightarrow q)] \Rightarrow q$

(b) $p \Rightarrow (q \wedge \neg q)$

(c) $(p \Rightarrow \neg q) \wedge (p \wedge q)$

(d) $[(p \vee q) \Rightarrow r] \wedge (r \Rightarrow \neg p)$.

When statement (b) is True, what can you conclude about $p$?

*Solution.*

(a) Truth table for $[p \wedge (p \Rightarrow q)] \Rightarrow q$:

| $p$ | $q$ | $[p \wedge (p \Rightarrow q)] \Rightarrow q$ |
|---|---|---|
| T | T | T |
| T | F | T |
| F | T | T |
| F | F | T |

(This is a tautology.)

(b) Truth table for $p \Rightarrow (q \wedge \neg q)$:

| $p$ | $q$ | $p \Rightarrow (q \wedge \neg q)$ |
|---|---|---|
| T | T | F |
| T | F | F |
| F | T | T |
| F | F | T |

If $p \Rightarrow (q \wedge \neg q)$ is True, we conclude that $p$ is False.

(c) Truth table for $(p \Rightarrow \neg q) \wedge (p \wedge q)$:

| $p$ | $q$ | $(p \Rightarrow \neg q) \wedge (p \wedge q)$ |
|---|---|---|
| T | T | F |
| T | F | F |
| F | T | F |
| F | F | F |

(This is a contradiction.)

(d) Truth table for $[(p \vee q) \Rightarrow r] \wedge (r \Rightarrow \neg p)$:

| $p$ | $q$ | $r$ | $[(p \vee q) \Rightarrow r] \wedge (r \Rightarrow \neg p)$ |
|---|---|---|---|
| T | T | T | F |
| T | T | F | F |
| T | F | T | F |
| T | F | F | F |
| F | T | T | T |
| F | T | F | F |
| F | F | T | T |
| F | F | F | T |

□

**2.2** (Converse and contrapositive).

(a) Suppose the following two statements are True:

- $p \Rightarrow q$
- $p$

i. Which row(s) of the truth table for implication corresponds to this case?

ii. What can we conclude about the truth value of $q$?

iii. Using sentences, give an example from your day-to-day life for when you can conclude something about a statement $q$ given that both $p$ and $p \Rightarrow q$ are True.

(b) Suppose the following two statements are True:

- $p \Rightarrow q$
- $q$

i. Which row(s) of the truth table for implication corresponds to this case?

ii. What can we conclude about the truth value of $p$?

iii. Using sentences, give an example from your day-to-day life for when you can't conclude something about a statement $p$ given that both $q$ and $p \Rightarrow q$ are True.

(c) Suppose the following two statements are True:

- $p \Rightarrow q$
- $\neg p$

i. Which row(s) of the truth table for implication corresponds to this case?

ii. What, if anything, can we conclude about the truth value of $q$?

iii. Using sentences, give an example from your day-to-day life for when you can't conclude something about a statement $q$ given that both $\neg p$ and $p \Rightarrow q$ are True.

(d) Suppose the following two statements are True:

- $p \Rightarrow q$
- $\neg q$

i. Which row(s) of the truth table for implication corresponds to this case?

ii. What, if anything, can we conclude about the truth value of $p$?

iii. Using sentences, give an example from your day-to-day life for when you can conclude something about a statement $p$ given that $q$ is False and $p \Rightarrow q$ is True.

*Solution.*

(a) Suppose $p \Rightarrow q$ and $p$ are True.

This is first row of the truth table for implication. Thus we can conclude $q$ is True. For example: **If it has rained, then the grass is wet.** Indeed if it has rained, then we can conclude the grass is wet.

(b) Suppose $p \Rightarrow q$ and $q$ are True.

This corresponds to either the first or third row of the truth table and so we can conclude nothing about $p$. For example: **If it has rained, then the ground is wet.** If we observe the grass is wet, we cannot be certain that it has rained. It may that a someone has recently watered the grass.

(c) Suppose $p \Rightarrow q$ and $\neg p$ are True.

This corresponds to either the third or fourth row of the truth table and so we cannot conclude anything about the truth value of $q$. For example: **If it has rained, then the ground is wet.** If it has not rained, then we cannot conclude anything about whether or not the ground is wet.

(d) Suppose $p \Rightarrow q$ and $\neg q$ are True.

This corresponds to the fourth row of the truth table and so we can conclude $p$ is False. For example: **If it has rained, then the ground is wet.** If the ground is not wet, then we can conclude that it has not rained. □

**2.3** (Tautology, contradiction, and logical equivalence)**.**

(a) Give an example of a compound statement that is a tautology. Give an example of a compound statement that is a contradiction. The terms are introduced in Definition 1.17 in the lectures.

(b) Determine if the following statement is a tautology: $[(p \vee q) \wedge (\neg p)] \Rightarrow q$.

(c) With an English sentence, explain what is meant by the notation $r \equiv s$.

(d) Show that $p \Rightarrow q \equiv (\neg p) \vee q$.

*Solution.*

(a) One finds an example of a tautology and of a contradiction in Tutorial Question 2.1.

(b) We find the truth table for the statement:

| $p$ | $q$ | $p \vee q$ | $\neg p$ | $(p \vee q) \wedge (\neg p)$ | $[(p \vee q) \wedge (\neg p)] \Rightarrow q$ |
|---|---|---|---|---|---|
| T | T | T | F | F | T |
| T | F | T | F | F | T |
| F | T | T | T | T | T |
| F | F | F | T | F | T |

As the statement is always True, it is a tautology.

(c) The notation $r \equiv s$ means that $r$ is logically equivalent to $s$.

(d) We compute the truth table for $(p \Rightarrow q) \Leftrightarrow [(\neg p) \vee q]$ and observe it is a tautology. Therefore $p \Rightarrow q \equiv (\neg p) \vee q$.

| $p$ | $q$ | $p \Rightarrow q$ | $\neg p$ | $(\neg p) \vee q$ | $(p \Rightarrow q) \Leftrightarrow [(\neg p) \vee q]$ |
|---|---|---|---|---|---|
| T | T | T | F | T | T |
| T | F | F | F | F | T |
| F | T | T | T | T | T |
| F | F | T | T | T | T |

□

**2.4** (Kaching!). Though the connectives we have introduced so far correspond to use in Australian English (not, or, and, etc...), we can use a truth table to define other connectives. For example, let \$ be the connective (that we will call *kaching!*) given by the following truth table:

| $p$ | $q$ | $p \,\$\, q$ |
|---|---|---|
| T | T | F |
| T | F | F |
| F | T | T |
| F | F | F |

(a) Is $p \,\$\, p$ a tautology? a contradiction? both? neither?

(b) Is the kaching! connective commutative? In other words, is $p \,\$\, q \equiv q \,\$\, p$?

(c) Which of the statements below are logically equivalent to $[(\neg p) \,\$\, p] \,\$\, [q \,\$\, (\neg q)]$:

- $\neg p \Rightarrow q$
- $p \vee (\neg q)$
- $\neg(p \vee q)$
- $\neg p$

*Solution.*

(a) We compute the truth table for $p \,\$\, p$.

| $p$ | $p \,\$\, p$ |
|---|---|
| T | F |
| F | F |

We observe $p \,\$\, p$ is a contradiction.

(b) The kaching! connective is not commutative: when $p$ is True and $q$ is False we have that $p \,\$\, q$ is False, but $q \,\$\, p$ is True. Therefore $p \,\$\, q \Leftrightarrow q \,\$\, p$ is not a tautology and so $p \,\$\, q \not\equiv q \,\$\, p$.

(c) Constructing the truth tables for each of the compound statements we find $(\neg p \,\$\, p) \,\$\, (q \,\$\, \neg q) \equiv \neg(p \vee q)$:

| $p$ | $q$ | $\neg p$ | $\neg p \Rightarrow q$ | $\neg q$ | $p \vee (\neg q)$ | $p \vee q$ | $\neg(p \vee q)$ |
|---|---|---|---|---|---|---|---|
| T | T | F | T | F | T | T | F |
| T | F | F | T | T | T | T | F |
| F | T | T | T | F | F | T | F |
| F | F | T | F | T | T | F | T |

| $p$ | $q$ | $(\neg p) \mathbin{\$} p$ | $q \mathbin{\$} (\neg q)$ | $[(\neg p) \mathbin{\$} p] \mathbin{\$} [q \mathbin{\$} (\neg q)]$ |
|---|---|---|---|---|
| T | T | T | F | F |
| T | F | T | T | F |
| F | T | F | F | F |
| F | F | F | T | T |

□

**Topics: condition vs statement, universal and existential quantifiers, English to logic and vice versa**

**2.5** (Statements and conditions)**.** For each expression below, determine if it is a statement or a condition.

- For each condition give the values of the domain for which it is True/False.
- For each statement, determine if it is True or False. Justify your response by writing a sentence.

(a) $2x + 1 \geqslant 3$, for $x \in \mathbf{R}$

(b) $(\exists x \in \mathbf{R})\ 2x + 1 \geqslant 3$

(c) $(\forall x \in \mathbf{R})\ 2x + 1 \geqslant 3$

(d) $x^2 < 0$, for $x \in \mathbf{R}$

(e) $(\exists x \in \mathbf{R})\ x^2 < 0$

(f) $(\forall x \in \mathbf{R})\ x^2 < 0$

(g) $xy = y$, for $x, y \in \mathbf{R}$

(h) $(\forall x, y \in \mathbf{R})\ xy = y$

(i) $(\exists x, y \in \mathbf{R})\ xy = y$

(j) $(\forall x \in \mathbf{R})[(\exists y \in \mathbf{R})\ xy = y]$

(k) $(\exists x \in \mathbf{R})[(\forall y \in \mathbf{R})\ xy = y]$.

*Solution.*

(a) This is a condition. It is True when $x \geqslant 1$.

(b) This statement is True. By part (a) we can find at least one value of $x$ so that $2x+1 \geqslant 3$.

(c) This statement is False. By part (a) there is at least one value of $x$ so that $2x + 1 < 3$.

(d) This is a condition. It is False for every $x \in \mathbf{R}$ as the square of any real number is non-negative.

(e) This is a statement. This is a False statement as we cannot find at least one real number whose square is negative. The square of every real number is non-negative.

(f) This is a statement. Since there exists at least one value of the domain for which the statement is False, e.g. $x = 1$, this statement is False.

(g) This is a condition. It is True whenever $x = 1$ or $y = 0$.

(h) This is a statement. It is False because we can find at least one pair $x, y \in \mathbf{R}$ so that $xy \neq y$. For example, $x = 2, y = 2$.

(i) This is a statement. It is True because we can find at least one pair $xy \in \mathbf{R}$ so that $xy = y$. For example, $x = 2, y = 0$.

(j) This is a statement. It is True because no matter which value of $x$ we consider taking $y = 0$ yields $xy = y$.

(k) This is a statement. It is True as taking $x = 1$ yields $xy = y$ for all $y \in \mathbf{R}$. □

**2.6** (Translating into the language of mathematics)**.** Translate the following sentences into formal logic. Which ones are True, which ones are False, and which ones are neither? If the sentence is ambiguous, explain why.

(a) "There is a rational number whose square is 7."

(b) "All even numbers are positive."

(c) "All subsets of the real numbers contain 12."

(d) "There are no real solutions of the equation $x^4 + 1 = 0$."

(e) "Every square root of 4 is positive."

(f) "The product of two integers is sometimes zero."

(g) "Some number is a multiple of both 6 and 10."

(h) "$a$ is a multiple of $n$."

*Solution.*

(a) $(\exists r \in \mathbf{Q})\ r^2 = 7$ False

(b) $(\forall n \in \mathbf{Z})\ 2n > 0$ False

(c) $(\forall S \subseteq \mathbf{R})\ 12 \in S$ False

(d) $\neg[(\exists x \in \mathbf{R})\ x^4 + 1 = 0]$ True

(e) $(\forall x \in \mathbf{R})\ (x^2 = 4 \Rightarrow x > 0)$ False

(f) $(\exists x, y \in \mathbf{Z})\ xy = 0$ True

(g) $(\exists n, k, \ell \in \mathbf{Z})\ (n = 6k \wedge n = 10\ell)$ True, e.g. $n = 30$

(h) $(\exists k \in \mathbf{Z})\ a = kn$ Depends on $a$ and $n$. □

**2.7** (Prime numbers)**.** A *prime number* is a natural number with exactly two positive divisors. For example: 7 is a prime number. (If $d$ is a divisor of 7, then $d = 1$ or $d = 7$.)

(a) Express the condition "$p$ is a prime number" using the language of formal logic. Use the notation $a \mid n$, which means "$a$ is a divisor of $n$".

(b) Let $P$ be the set of prime numbers. Express the condition "every integer greater than 1 that is not prime has at least one prime divisor" using the language of formal logic.

*Solution.*

(a) $(p \in \mathbf{N}) \wedge \#\{d \in \mathbf{N} :\ d \mid p\} = 2.$

If we don't want to use set cardinality, it gets a bit more involved:

$$(p \in \mathbf{N}) \wedge (p > 1) \wedge [(d \mid p) \Rightarrow (d = 1 \vee d = p)].$$

(b) $(\forall n \in \mathbf{Z})\ [(n > 1 \wedge n \notin P) \Rightarrow ((\exists p \in P)\ p \mid n)].$

(Note that this statement remains True even if we leave out the condition $n \notin P$.) □

**2.8.** Negate each of these statements, then pull the negation as far to the right as possible. (If you get stuck, first write the statement as an English sentence.)

(a) $(\forall x \in S)\ \ q(x)$

(b) $(\forall x \in S)\ \ [(\exists y \in S)\ \ p(x, y)]$

(c) $(\exists x \in S)\ \ [(\forall y \in S)\ \ p(x, y)]$

(d) $(\exists x \in S)\ \ (q(x) \Rightarrow r(x))$.

Let $p(x, y)$ be the statement "$x + y = 0$", and let $S = \mathbf{R}$. Are the non-negated versions of (b) and (c) saying the same thing? Are they both True?

*Solution.*

(a) $(\exists x \in S)\ \neg q(x)$

(b) $(\exists x \in S)\ [(\forall y \in S)\ \neg p(x, y)]$

(c) $(\forall x \in S)\ [(\exists y \in S)\ \ \neg p(x, y)]$

(d) $(\forall x \in S)\ (q(x) \wedge \neg r(x))$.

(b) and (c) are saying **different** things. (b) is True (since every real number has an additive inverse) and (c) is False (there is no real number that is the additive inverse of everything else). □

**2.9** (Challenge: one connective to rule them all)**.** It turns out that one can express all the connectives in the set $\{\neg, \wedge, \Rightarrow, \vee, \Leftrightarrow\}$ in terms of the smaller set of connectives $\{\neg, \vee\}$.

(a) Show $p \wedge q \equiv \neg(\neg p \vee \neg q)$.

(b) Show $p \Rightarrow q \equiv (\neg p) \vee q$.

(c) Show $p \Leftrightarrow q \equiv (p \Rightarrow q) \wedge (q \Rightarrow p)$.

(d) Find a statement that uses only $\neg$ and $\vee$ and is logically equivalent to $p \Leftrightarrow q$.

This motivates the following definition: Let $C$ be a set of connectives. We say that $C$ is *functionally complete* when every connective in the set $\{\neg, \wedge, \Rightarrow, \vee, \Leftrightarrow\}$ is logically equivalent to a compound statement that uses only connectives in the set $C$.

Our work in (a)–(d) shows that $\{\neg, \vee\}$ is functionally complete.

(e) Determine if there is a functionally complete set $\{\star\}$ containing a single element.

[**Hint**: Try to build the truth table for $\star$. You win if you can express each of $\neg$ and $\vee$ in terms of $\star$. A reasonable guess would be that $p \star p \equiv \neg p$, so start with this.]

| $p$ | $q$ | $p \star q$ |
|---|---|---|
| T | T | ? |
| T | F | ? |
| F | T | ? |
| F | F | ? |

*Solution.*

(a) We could build a truth table to check this, but another approach is to use Example 1.19 and $\neg(\neg p) \equiv p$:

$$\neg(\neg p \vee \neg q) \equiv \neg(\neg p) \wedge \neg(\neg q) \equiv p \wedge q.$$

(b) Via truth table:

| $p$ | $q$ | $\neg p$ | $(\neg p) \vee q$ | $p \Rightarrow q$ |
|---|---|---|---|---|
| T | T | F | T | T |
| T | F | F | F | F |
| F | T | T | T | T |
| F | F | T | T | T |

(c) Via truth table:

| $p$ | $q$ | $p \Rightarrow q$ | $q \Rightarrow p$ | $(p \Rightarrow q) \wedge (q \Rightarrow p)$ | $p \Leftrightarrow q$ |
|---|---|---|---|---|---|
| T | T | T | T | T | T |
| T | F | F | T | F | F |
| F | T | T | F | F | F |
| F | F | T | T | T | T |

(d) Combine (b) and (c) to get:

$$p \Leftrightarrow q \equiv [(\neg p) \vee q] \wedge [(\neg q) \vee p].$$

(e) If $p \star p \equiv \neg p$, then we must have:

| $p$ | $q$ | $p \star q$ |
|---|---|---|
| T | T | F |
| T | F | ? |
| F | T | ? |
| F | F | T |

This leaves four possibilities for filling in the remaining question marks.

We notice that the two rows that are already filled in are precisely the opposites of the two corresponding rows in the truth table of $p \vee q$. So we guess that $p \star q \equiv \neg(p \vee q)$:

| $p$ | $q$ | $p \star q$ |
|---|---|---|
| T | T | F |
| T | F | F |
| F | T | F |
| F | F | T |

With this truth table, we have

$$\neg p \equiv p \star p \qquad \text{and} \qquad p \vee q \equiv \neg(p \star q) \equiv (p \star q) \star (p \star q). \qquad \square$$