Topics: set operations, relations

4.1 (Relations on sets)**.** Let $A = \{1, 2, 3\}$ and $B = \{1, 3\}$. Which of the following statements are True? For each, write a sentence that refers to the appropriate definition to justify your response.

(a) $A = B$

(b) $A \subsetneq B$

(c) $B \subseteq A$

(d) $(1, 2) \in A \times B$

(e) $(1, 2) \in B \times A$

(f) $A \cup B$ contains five elements

(g) $A \cup B$ contains three elements

(h) $A \cap B \neq \varnothing$.

Solution.

(a) False. A and B do not have the same elements.

(b) False. A has an element that B does not.

(c) True. Every element of B is an element of A.

(d) False. $2 \notin B$.

(e) True. $1 \in B$ and $2 \in A$.

(f) False. $A \cup B = \{1, 2, 3\}$.

(g) True. $A \cup B = \{1, 2, 3\}$.

(h) True. $A \cap B = \{1, 3\}$. □

4.2 (Quantified statements about sets)**.** Let $A = \{1, 4, 7, 25\}$ and $B = \{2, 7, 60\}$. Which of the following are True?

(a) $(\forall a \in A)\ \ [(\exists b \in B)\ \ b + a \text{ is even}]$

(b) $(\exists a \in A)\ \ [(\forall b \in B)\ \ a > b]$.

Solution.

(a) True, because B contains both even and odd numbers.

(b) False, because $60 \in B$ is larger than every element of A. □

4.3 (Proofs about sets)**.** Using the definitions of the notations $\subseteq$, $=$, $\cup$, and $\cap$, give a proof of each of the following theorems. To think about how to get started, look back at the proof of Theorems 2.19 and 2.20.

You may use either an informal proof style (with more words and sentences) or a formal one (with mostly symbols and logic notation), but try to keep to one logical step per line, and check that each step follows from the previous ones before moving on.

For the "if and only if" theorems, recall that $p \Leftrightarrow q \equiv [(p \Rightarrow q) \wedge (q \Rightarrow p)]$.

(At first glance some of you may think that the theorems below are **obvious**. If indeed they are **obvious** you should have little trouble writing down a proof that would convince a skeptical peer.)

(a) "Let A, B and C be sets. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$."

Proof. Let A, B and C be sets so that $A \subseteq B$ and $B \subseteq C$. Let $x \in A$.
.........(Fill in the missing steps) ....
We conclude that $A \subseteq C$. □

(b) "Let A and B be sets. We have $B \subseteq A$ if and only if $A \cap B = B$."

Proof. In one direction, let A and B be sets so that $B \subseteq A$.
.........(Fill in the missing steps) ....
Therefore $A \cap B = B$.

For the other direction, let A and B be sets so that $A \cap B = B$.
.........(Fill in the missing steps) ....
Therefore $B \subseteq A$. □

(c) "Let A and B be sets. We have $A \cup B = B$ if and only if $A \subseteq B$."
(Proceed similarly to the previous part.)

(d) Using parts (b) and (c), what can you say about A and B when $A \cap B = A \cup B$?

(e) Give a counterexample to show that the following statement is False in general:

$$A \cup (B \cap C) = (A \cup B) \cap C.$$

Solution.

(a) Let A, B and C be sets so that $A \subseteq B$ and $B \subseteq C$. Let $x \in A$.

i. Since $A \subseteq B$ and $x \in A$, we have $x \in B$. (premise, def of $\subseteq$)
ii. Since $B \subseteq C$ and $x \in B$, we have $x \in C$. (premise, i., def of $\subseteq$)

We conclude that $A \subseteq C$.

(b) In one direction: let A and B be sets so that $B \subseteq A$.
Let $x \in B$.

i. Since $B \subseteq A$, we have $x \in A$. (def of $\subseteq$)
ii. Therefore $x \in A \cap B$. (def of $\cap$)

So $B \subseteq A \cap B$.
Now let $y \in A \cap B$.

i. Then $y \in B$. (def of $\cap$)

So $A \cap B \subseteq B$.
From $B \subseteq A \cap B$ and $A \cap B \subseteq B$, we conclude that $A \cap B = B$.

For the other direction, let A and B be sets so that $A \cap B = B$.
Let $x \in B$.

i. Since $A \cap B = B$ by assumption, we have $x \in A \cap B$.

ii. So $x \in A$. (def of $\cap$)

We conclude that $B \subseteq A$.

(c) Proceed similarly to the previous part.

(d) Suppose $A \cup B = A \cap B$. Since $B \subseteq A \cup B$ and $A \cap B \subseteq B$, we have

$$B \subseteq A \cup B = A \cap B \subseteq B.$$

Since the leftmost and rightmost parts of the above are $B = B$, it means that all the inclusions are equalities:

$$B = A \cup B = A \cap B = B.$$

From $B = A \cup B$ and part (c), we know that $A \subseteq B$. From $B = A \cap B$ and part (b), we know that $B \subseteq A$.

Therefore $A = B$.

(e) For example, we can take $A = \{1, 2\} = B$ and $C = \varnothing$. Then

$$A \cup (B \cap C) = \{1, 2\}$$
$$(A \cup B) \cap C = \varnothing.$$

□

4.5 (Relations on sets with two elements)**.** Let $S = \{a, b\}$ be a set with two elements.

(a) Find all possible relations on S.

(b) Which of the above relations are orders on S?

(c) Which of the above relations are functions $S \longrightarrow S$?

Solution.

(a) A relation on S is a subset of $S \times S$. There are 16 of them!

(b) The only orders on S are $\{(a, b)\}$ (corresponding to $a < b$) and $\{(b, a)\}$ (corresponding to $b < a$).

(c) The functions are $\{(a, a), (b, a)\}$, $\{(a, a), (b, b)\}$, $\{(a, b), (b, a)\}$, and $\{(a, b), (b, b)\}$. Write them as functions in the usual notation! □

4.5 (Ordered tuples)**.** Let A_1 and A_2 be sets. As we have seen in Definition 2.15, an *ordered pair* (a_1, a_2) is a set of the form

$$(a_1, a_2) = \big\{\{a_1\}, \{a_1, a_2\}\big\}, \qquad \text{where } a_1 \in A_1 \text{ and } a_2 \in A_2.$$

(This definition is due to the Polish mathematician Kazimierz Kuratowski.)

(a) Prove that $(a_1, a_2) = (b_1, b_2)$ if and only if $a_1 = b_1$ and $a_2 = b_2$.

(b) Based on the above, give a recursive definition of the n-tuple $(a_1, a_2, \dots, a_n)$ using ordered pairs. Prove the property

$$(a_1, \dots, a_n) = (b_1, \dots, b_n) \Leftrightarrow a_i = b_i \text{ for all } i = 1, \dots, n.$$

Solution.

(a) **(⇒)** Suppose

$$(a_1, a_2) = (b_1, b_2).$$

Then

$$\big\{\{a_1\}, \{a_1, a_2\}\big\} = \big\{\{b_1\}, \{b_1, b_2\}\big\}.$$

Since the two sets are equal, their elements are the same. In particular,

$$\{a_1\} \in \big\{\{b_1\}, \{b_1, b_2\}\big\}.$$

Hence either $\{a_1\} = \{b_1\}$ or $\{a_1\} = \{b_1, b_2\}$.

Case 1. If $\{a_1\} = \{b_1\}$, then $a_1 = b_1$. Since the outer sets are equal, the remaining elements must also be equal: $\{a_1, a_2\} = \{b_1, b_2\}$. Substituting $a_1 = b_1$ gives

$$\{a_1, a_2\} = \{a_1, b_2\},$$

which implies $a_2 = b_2$.

Case 2. If $\{a_1\} = \{b_1, b_2\}$, then $\{b_1, b_2\}$ has only one element and $a_1 = b_1 = b_2$. From the equality of the outer sets we again obtain $\{a_1, a_2\} = \{b_1\}$, thus $a_1 = a_2 = b_1 = b_2$. In particular, $a_1 = b_1$ and $a_2 = b_2$.

(⇐) Conversely, if $a_1 = b_1$ and $a_2 = b_2$, then

$$\{a_1\} = \{b_1\}, \qquad \{a_1, a_2\} = \{b_1, b_2\}.$$

Hence

$$\big\{\{a_1\}, \{a_1, a_2\}\big\} = \big\{\{b_1\}, \{b_1, b_2\}\big\},$$

so $(a_1, a_2) = (b_1, b_2)$.

(b) Let $n \geqslant 2$ be an integer, let $A_1, \ldots, A_n$ be sets, and let $a_1 \in A_1, \ldots, a_n \in A_n$ be elements. The *n-tuple* $(a_1, a_2, \ldots, a_n)$ is defined recursively using ordered pairs as follows:

$$(a_1, a_2) := \big\{\{a_1\}, \{a_1, a_2\}\big\},$$

and for $n \geqslant 3$,

$$(a_1, a_2, \ldots, a_n) := \big(a_1, (a_2, \ldots, a_n)\big).$$

We prove the statement by induction on n.

Base case ($n = 2$). This is proved in (a).

Inductive step. Assume the statement holds for $(n-1)$-tuples. That is,

$$(a_2, \ldots, a_n) = (b_2, \ldots, b_n) \quad \Longleftrightarrow \quad a_i = b_i \text{ for } i = 2, \ldots, n.$$

By definition,

$$(a_1, \ldots, a_n) = (b_1, \ldots, b_n)$$

means

$$\big(a_1, (a_2, \ldots, a_n)\big) = \big(b_1, (b_2, \ldots, b_n)\big).$$

By the property of ordered pairs from part (a), this implies

$$a_1 = b_1 \quad \text{and} \quad (a_2, \ldots, a_n) = (b_2, \ldots, b_n).$$

By the induction hypothesis,

$$a_i = b_i \quad \text{for } i = 2, \ldots, n.$$

Thus $a_i = b_i$ for all $i = 1, \ldots, n$.

Conversely, if $a_i = b_i$ for all $i = 1, \ldots, n$, then in particular

$$a_1 = b_1 \quad \text{and} \quad (a_2, \ldots, a_n) = (b_2, \ldots, b_n).$$

Hence

$$\big(a_1, (a_2, \ldots, a_n)\big) = \big(b_1, (b_2, \ldots, b_n)\big),$$

so

$$(a_1, \ldots, a_n) = (b_1, \ldots, b_n). \qquad \square$$

4.6 (Regularity rules out self-containment)**.** Among the ZFC set axioms we have the Axiom of Regularity: if A is a non-empty set, there exists $y \in A$ such that $A \cap y = \varnothing$.

Use a proof by contradiction to show that there does not exist any set S with the property that $S \in S$.

[**Hint**: Take $A = \{S\}$ in the Axiom of Regularity and find a contradiction.]

Solution. We proceed by contradiction.

Suppose such a set S exists and let $A = \{S\}$. Then A is non-empty, so by the Axiom of Regularity there exists $y \in A$ such that $A \cap y = \varnothing$.

But A contains a single element, namely S. So we must have $y = S$, hence $A \cap S = \varnothing$.

However, we have assumed that $S \in S$, and clearly $S \in A$, so $S \in A \cap S$, contradiction. $\square$

Topics: bounds, supremum, infimum, order

4.7 (Finding a supremum)**.** Consider the set

$$A = \left\{1 - \frac{1}{n} : \ n \in \mathbf{N}, n \neq 0\right\}.$$

(a) Draw a picture of A on the real line.

(b) Using your picture, determine for which values of β we can say that β is an upper bound for A in $\mathbf{R}$.

(c) Based on your answer to (b), what do you think is the supremum of A in $\mathbf{R}$?

(d) Suppose that your answer to (c) is incorrect. This means there is an upper bound r that is smaller than the one you have conjectured in (c). Prove that there is an element of A that is bigger than r. At some point in your proof, you might need to use the following version of the Archimedean Principle (see Exercise 2.30):

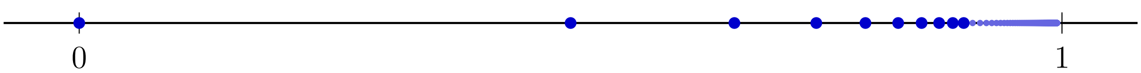
$$(\forall \varepsilon > 0) \ \ [(\exists n \in \mathbf{N}) \ 1/n < \varepsilon].$$

[**Hint**: It might help to draw a picture. Based on the picture, what do you want to show?]

(e) Does your work in the previous part convince you that your answer in part (c) is correct? Take a moment to refer back to Definition 2.46 to be certain.

Solution.

(a) Here is an illustration of the first 200 elements of the set A (the first 10 are drawn a little bigger, the others smaller for better visibility):



(b) $\beta \geqslant 1$.

(c) $\beta = 1$.

(d) Assume $r < 1$. Let $d = 1 - r$. There exists $n \in \mathbf{N}$ so that $\frac{1}{n} < d$. Therefore

$$r = 1 - d < 1 - \frac{1}{n} < 1.$$

Since $1 - \frac{1}{n} \in A$ and $r < 1 - \frac{1}{n}$, r is not an upper bound for A in $\mathbf{R}$.

(e) Yes. From (d), we know that an upper bound for A must be $\geqslant 1$. Since 1 is an upper bound for A, 1 is the least upper bound of A. That is, 1 is the supremum of A. □

4.8 (Inequalities and squares)**.** Let $a, b \in \mathbf{R}_{>0}$. Using the axioms of the order relation on $\mathbf{R}$ and/or results we have proved about this order relation, prove the following:

(a) If $a > b$, then $a^2 > b^2$.

(b) If $a^2 \leqslant b^2$, then $a \leqslant b$.

(c) Give a counterexample to the statement in part (a) if we do not require a and b to be positive.

Solution.

(a) Since $a > 0$, by Theorem 2.41 we can multiply both sides of the inequality $a > b$ by a and get

$$a^2 > ab.$$

On the other hand, since $b > 0$, we can multiply both sides of the inequality $a > b$ by b and get

$$ab > b^2.$$

The claim now follows from Axiom (RO2).

(b) By Axiom (RO1), the statement in part (b) is simply the contrapositive of the statement in part (a), so we are done by part (a).

(c) Let $a = 1$, $b = -2$, then $a > b$ is True but $a^2 = 1 > 4 = b^2$ is False. ☐

4.9 (Minima and maxima)**.** Let S be an ordered set and A a subset of S. Consider the following definitions:

- We say that an element $M \in A$ is a *maximum* of A when $(\forall x \in A)\ x \leqslant M$.
- We say that an element $m \in A$ is a *minimum* of A when $(\forall x \in A)\ m \leqslant x$.

(a) Given $M \in A$, prove that M is a maximum of A if and only if (A has a supremum in S and $M = \sup A$).

(There is a similar result for minimum and infimum.)

Find, if they exist, the supremum, infimum, maximum, and minimum of each subset A of $\mathbf{R}$ given below.

Try to justify your answers. You may need to refer to Tutorial Question 4.8 or Theorem 2.60 for some of the parts. You may also assume that π and $\sqrt{3}$ are both irrational.

(b) $A = \{x \in \mathbf{R} :\ x^2 \leqslant 3\}$

(c) $A = \{x \in \mathbf{Q} :\ x^2 \leqslant 3\}$

(d) $A = \{x \in \mathbf{R} :\ x \leqslant \pi\}$

(e) $A = \{x \in \mathbf{Q} :\ x \leqslant \pi\}$.

Solution.

(a) **(⇒)** Suppose M is a maximum of A. Then by definition, for any element $x \in A$, we have $x \leqslant M$. Thus, M is an upper bound of A.

Let β be any upper bound of A. Since $M \in A$, we must have $M \leqslant \beta$. Therefore M is the least upper bound of A.

(⇐) Suppose A has a supremum in S and $M = \sup A$. Since M is the supremum, it is an upper bound of A, so

$$(\forall x \in A)\ x \leqslant M.$$

As $M \in A$, then by definition M is a maximum of A.

(b) I claim that $\sqrt{3}$ is a maximum of A. First note that $\sqrt{3} \in A$ since certainly $(\sqrt{3})^2 \leqslant 3$.

Let $x \in A$. If $x \leqslant 0$, then certainly $x \leqslant \sqrt{3}$. If $x > 0$, then by Tutorial Question 4.8 (b) we see that $x^2 \leqslant 3$ implies $x \leqslant \sqrt{3}$.

Therefore $\sqrt{3}$ is indeed a maximum of A, so by part (a) we also get that $\sup A = \sqrt{3}$.

A similar argument shows that $-\sqrt{3}$ is a minimum of A and $\inf A = -\sqrt{3}$.

(c) The same argument as in part (b) shows that $\sqrt{3}$ is an upper bound for A in $\mathbf{R}$.

A is certainly non-empty as $1 \in A$, so since it is bounded above, it has a supremum by the Completeness Axiom. Let $\beta = \sup A$, then $1 \leqslant \beta \leqslant \sqrt{3}$.

I claim that $\beta = \sqrt{3}$. We prove this by contradiction: suppose $\beta < \sqrt{3}$, then applying Theorem 2.60 we get some $r \in \mathbf{Q}$ such that $1 \leqslant \beta < r < \sqrt{3}$. By Tutorial Question 4.8 (a) applied to $\sqrt{3} > r$ we get $3 > r^2$, so $r \in A$, but then $\beta < r$ contradicts the fact that $\beta = \sup A$.

So $\sup A = \sqrt{3}$. However, A does not have a maximum (if it did, by part (a) it would have to be $\sqrt{3}$, but $\sqrt{3} \notin A$ as $\sqrt{3}$ is irrational).

In a similar manner we see that $\inf A = -\sqrt{3}$ and that A has no minimum.

(d) It is clear from the definition of A that π is an upper bound for A and that $\pi \in A$, so π is a maximum of A, hence also $\sup A = \pi$.

It is even more clear from the definition that A is not bounded below, so it has neither an infimum nor a minimum.

(e) Now we have that π is an upper bound for A. A similar argument to part (c) shows that $\sup A = \pi$. But A does not have a maximum (if it did, it would have to be π, but π is irrational).

Again, A does not have an infimum or a minimum. □

4.10 (Well-Ordering Property of **N**). The objective is to prove Theorem 2.21: Every non-empty subset $S \subseteq \mathbf{N}$ has a *minimum*: there exists $m \in S$ such that $m \leqslant x$ for all $x \in S$.

Start by defining the following subset of **N**:

$$A = \{k \in \mathbf{N} : \text{ every } T \subseteq \mathbf{N} \text{ that contains an element } \leqslant k \text{ has a minimum}\}.$$

(a) Convince yourself (and the person next to you, for good measure) that if we prove $A = \mathbf{N}$, then the statement of the Theorem is proved.

(b) Use induction to prove that $(\forall n \in \mathbf{N})\ n \in A$, therefore $A = \mathbf{N}$.

Solution.

(a) We show that if $A = \mathbf{N}$ then the Theorem holds.

Let S be a non-empty subset of **N**. Since S is a non-empty, it contains some natural number n. Since $A = \mathbf{N}$, we have $n \in A$, which means that every $T \subseteq \mathbf{N}$ that contains an element $\leqslant n$ has a minimum. But S is such a subset, so S has a minimum.

(b) **Base case:** Is $0 \in A$?

Let $T \subseteq \mathbf{N}$ be a subset that contains an element $\leqslant 0$. Then $0 \in T$, so T clearly has 0 as a minimum (since **N** has 0 as a minimum).

Induction step: Fix $k \in \mathbf{N}$ and suppose that $k \in A$. Is $(k+1) \in A$?

Let $T \subseteq \mathbf{N}$ be a subset that contains an element $\leqslant k+1$. We have two cases:

- T contains an element $\leqslant k$. Then by the induction hypothesis, T has a minimum.
- T does not contain any elements $\leqslant k$. But T contains an element $\leqslant k+1$, so we must have $(k+1) \in T$, and $k+1$ is a minimum of T.

In both cases we conclude that S has a minimum, so $(k+1) \in A$. □